APR 14 '60
MAY 2 '60

NEW YORK UNIVERSITY
WASHINGTON SQUARE COLLEGE OF ARTS AND SCIENCE
MATHEMATICS RESEARCH GROUP

RESEARCH REPORT No. EM-47

TECHNIQUES OF APPLIED MATHEMATICS

THEORY OF DISTRIBUTIONS

by

BERNARD FRIEDMAN

CONTRACT No. AF-19(122)-42

New York University
Washington Square College of Arts and Science
Mathematics Research Group

Research Report No. EM-47

TECHNIQUES OF APPLIED MATHEMATICS
THEORY OF DISTRIBUTIONS

by

BERNARD FRIEDMAN

Bernard Friedman
Bernard Friedman

Morris Kline
Morris Kline
Project Director

The research reported in this document has been made possible through support and sponsorship extended by the Geophysics Research Division of the Air Force Cambridge Research Center, under Contract No. AF-19(122)-42. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

October, 1952

NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY

Manufactured in the United States for New York University Press
by the University's Office of Publications and Printing

Abstract

This report is the second chapter of a proposed book on the Techniques of Applied Mathematics. Because of the time lag before publication and because of the interest in the subject matter, it was thought advisable to issue selected chapters as project reports. The first chapter reviews well-known facts concerning linear algebra and will not be issued as a report.

This second chapter discusses the theory of the Dirac delta function and singular functions which play such an important part in classical applied mathematics and in modern physics, especially quantum theory. Although the results are in most cases well-known, it has been thought worthwhile to present a fairly rigorous mathematical treatment of these topics in order to give the physicists confidence in the use of the delta function. The treatment is based on the theory of distributions which was developed by Laurent Schwartz.

Comments and suggestions by readers of this report will be most welcome to the author.

Table of Contents

	Page
Abstract	1
Introduction	72
Translations of Distributions	75
Differentiation	76
Examples of Derivatives of Distributions	78
Functions with Infinite Jumps	80
Multiplication of Distributions	83
Convergence of Distributions	85
Values of Distributions and Change of Variables	91
Integration	93
Convolutions or Folding of Distributions	93
Distribution in More than One Dimension	95
δ-Functions in Different Coordinate Systems	97
Fourier Integrals	100
Properties of the Fourier Transforms	102
Fourier Transform in Three Dimensions	103
Evaluating Finite Part Integrals	105
Principal Value Integrals and Singular Integral Equations	110
Summary	113

Introduction

A good part of mathematical physics is devoted to a discussion of the singularities in a field; for example, point sources, line sources, dipoles, electric and magnetic currents, etc. All these phenomena represent some kind of discontinuity in the field and it is difficult to give an adequate mathematical description of them.

The physicists have tried to simplify the mathematics of such phenomena by introducing the Dirac δ-function. Its usual definition is this: the δ-function is zero for every value of x except the origin, where it is infinite in such a way that

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

Mathematically, this definition is nonsense. If a function is zero everywhere except at one point, its integral, be it Riemann, Lebesgue, or anyone else's, is necessarily equal to zero. However, the δ-function has proved itself to be of such great utility that the physicist rightly refuses to give it up. It is the mathematician's task to find a method by which the use of the δ-function can be justified.

One such method has been known for a long time. It is to consider the δ-function as a disguised Stieltjes integral. This idea, although correct, has not proved very fruitful. Recently, Laurent Schwartz* has proposed a method which not only justifies the use of the δ-function but also the use of all derivatives of the δ-function. By this method every continuous function can be differentiated arbitrarily often. All the stop signals that the mathematicians have imposed upon the free handling of mathematics, such as questions about convergence and the interchange of limiting operations, become much easier to avoid and to neglect. The method, called the Theory of Distributions by its discoverer, will be developed in this chapter.

Interpretation of the δ-function

What should be understood by the δ-function? Its most important property and the one that makes it so useful is the following:

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0) \quad ; \tag{2.1}$$

this property may also be expressed thus: given a continuous function $\phi(x)$, the δ-function picks out the value of that function at the origin. We may say then that

* "Theorie des Distributions et Tranformation de Fourier," Annales de l'Universite de Grenoble, tome XXI, pp. 57-74, 1945; and "Generalisation de la Notion de Fonction ... de Fourier," tome XXIII, pp. 7-24, 1947-8. See also L. Schwartz, Theorie des Distributions, Actualites Scientifiques et Industrielles No. 1091 and 1122, Paris, 1950-1.

the δ-function is really a functional on the space of continuous functions because it assigns a number to every continous function.

This viewpoint may appear more plausible if we consider that every integrable function $f(x)$ may be interpreted as a linear functional. Let $\mathcal{C}$ be the linear vector space of continuous functions $\phi(x)$ which vanish outside a finite interval. The f-functional assigns to any function $\phi(x)$ in the space $\mathcal{C}$ the value

$$f(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx. \tag{2.2}$$

This equation shows that any integrable function $f(x)$ defines a linear functional on $\mathcal{C}$. Conversely, suppose there is given a linear functional on $\mathcal{C}$; that is, to any $\phi(x)$ in $\mathcal{C}$ let there be assigned a number denoted by $f(\phi)$ such that

$$f(\alpha\phi_1 + \beta\phi_2) = \alpha f(\phi_1) + \beta f(\phi_2), \tag{2.3}$$

where α and β are arbitrary numbers. In general, there will not exist a function $f(x)$ such that (2.2) is valid. However, if $f(\phi)$ is also a continuous functional, that is, if the values $f(\phi_n)$ converge to zero as the maximum of $\phi_n(x)$ converges to zero, then as F. Riesz has proved, there exists a function $m(x)$ such that $f(\phi)$ can be written as a Stieltjes integral:

$$f(\phi) = \int_{-\infty}^{\infty} \phi(x)dm(x). \tag{2.4}$$

In case $m(x)$ has a derivative $f(x)$ and is equal to the integral of its derivative, equation (2.4) becomes

$$f(\phi) = \int_{-\infty}^{\infty} \phi(x)f(x)dx.$$

Now, let us consider the δ-functional: To every $\phi(x)$ in $\mathcal{C}$ it assigns the number $\phi(0)$. This definition clearly satisfies (2.3) and thus shows that the δ-functional is linear. The δ-functional also is continuous, because if the maximum of $\phi_n(x)$ converges to zero, then surely the values $\phi_n(0)$ converge to zero. By the theorem of F. Riesz there must exist a function $m(x)$ such that

$$\delta(\phi) = \phi(0) = \int_{-\infty}^{\infty} \phi(x)dm(x) .$$

This function, $m(x)$, is actually the Heaviside unit function, $H(x)$, defined as follows:

$$H(x) = 0, \qquad x < 0$$
$$= 1, \qquad x > 0 ;$$

therefore

$$\delta(\phi) = \int_{-\infty}^{\infty} \phi(x)dH(x). \tag{2.5}$$

Of course, equation (2.5) and the interpretation of the δ-function as a Stieltjes integral were well known. The contribution of Laurent Schwartz consists in the

introduction of the concept of functionals and in the investigation of the implications of that concept.

Instead of continuing the discussion of functionals on the space $\mathcal{C}$, we shall introduce a space $\mathcal{D}$, which will be more useful in applications. Let $\mathcal{D}$ be the linear vector space of functions $\phi(x)$ which have continuous derivatives of every order and which vanish identically outside a finite interval. Any such function $\phi(x)$ will be called a testing function. The use of the space $\mathcal{D}$ will enable us to differentiate functionals as often as we please.

Let $T(\phi)$ denote the number a functional T assigns to any testing function $\phi(x)$. A functional is continuous if the numbers $T(\phi_n)$ converge to zero whenever the sequence of testing functions $\phi_n(x)$ converges to zero in the sense of $\mathcal{D}$. A sequence of testing functions $\phi_n(x)$ converges to zero in the sense of $\mathcal{D}$ if the maximum of $\frac{d^p\phi_n}{dx^p}$ for p fixed ($p = 0,1,2,\ldots$) converges to zero as n becomes infinite and if all the functions $\phi_n(x)$ are zero outside the same finite interval. It is an immediate consequence of this definition that if ϕ_n converge to ϕ in the sense of $\mathcal{D}$ (i.e., $\phi_n - \phi$ converge to zero in the sense of $\mathcal{D}$), then the numbers $T(\phi_n)$ converge to $T(\phi)$. Any linear continuous functional, T, on the space $\mathcal{D}$ will be called a distribution..

The set of all distributions forms a linear vector space which we denote by $\mathcal{D}'$. This space is said to be the space dual to $\mathcal{D}$. Any integrable function $f(x)$ defines a distribution by the formula:

$$f(\phi) = \int f(x)\phi(x)dx. \tag{2.6}$$

(Note: all integrals will be from $-\infty$ to $+\infty$, unless otherwise specified.) In such a case, we say the distribution is a function. Other examples of distributions are the δ-functional, which assigns the value $\phi(0)$ to every function $\phi(x)$, and the functionals which assign the value $\phi^{(p)}(0)$ to $\phi(x)$.

The concept of distribution is a generalization of the concept of function. A function is defined by the values it assumes for given values of x. A distribution does not have any value for a given value of x. It is defined only by the values it assigns to the set of testing functions. Nevertheless, because the function notation is so convenient to use, we associate a symbolic function, $t(x)$, with every distribution T, and write

$$T(\phi) = \int t(x)\phi(x)dx. \tag{2.7}$$

The function $t(x)$ and the integral in (2.7) will have no meaning except that given by the left side of (2.7). However, if T is defined by an integrable function $f(x)$, as in (2.6), we shall identify $t(x)$ with $f(x)$.

With the δ-functional we associate the δ-function, $\delta(x)$, which is a symbolic function and has no values or meaning other than what is implied by the formula

$$\phi(0) = \int \delta(x)\phi(x)dx.$$

Note that since a distribution is a continuous functional,

$$\lim T(\phi_n) = T(\lim \phi_n),$$

where the $\lim \phi_n$ is to be understood in the sense of convergence in $\mathcal{D}$. In terms of symbolic functions this equation can be expressed as follows:

Theorem I. If $t(x)$ is a symbolic function and $\phi_n(x)$ is a sequence of testing functions converging in $\mathcal{D}$, then

$$\lim \int t(x)\phi_n(x)dx = \int t(x) \lim \phi_n(x)dx.$$

Problems

2.1 Prove the statement in the text that an integrable function $f(x)$ defines a distribution by the formula (2.6). Hint. Show that $\int f(x)\phi(x)dx$ is a linear continuous functional on $\mathcal{D}$.

2.2 Show that the function

$$\phi(x) = e^{\frac{1}{x^2} - \frac{1}{(x-a)^2}} \quad , 0 < x < a$$
$$= 0 \quad , x \leq 0 \text{ , or } a \leq x$$

is a testing function.

Translations of Distributions.

We shall now develop a calculus of distributions and their associated symbolic functions. Our aim is to extend to distributions the ideas and concepts which are valid for ordinary functions. This extension is usually obtained by re-writing these ideas and concepts so that they apply to functionals.

Consider the concept of the translation of a distribution. Suppose the distribution is defined by a function $f(x)$; then the translation of it through a distance h is $f(x+h)$. and the corresponding functional is

$$\int f(x+h)\phi(x)dx = \int f(x)\phi(x-h)dx. \tag{2.8}$$

Write $\tau_h f$ for the translation of $f(x)$ through a distance h: then (2.8) becomes

$$\tau_h f(\phi) = f(\tau_{-h}\phi) . \tag{2.9}$$

Equation (2.9) is used to define the translation of a distribution. Let T be an arbitrary distribution and let $\tau_h T$ be its translation through an amount h: then we define

$$\tau_h T(\phi) = T(\tau_{-h}\phi). \tag{2.10}$$

The corresponding symbolic function will be written $t(x+h)$, and from (2.10) we have

$$\int t(x+h)\phi(x)dx = \int t(x)\phi(x-h)dx;$$

thus $t(x)$ actually behaves like a function.

Suppose T is the δ-functional; then

$$\int \delta(x+h)\phi(x)dx = \int \delta(x)\phi(x-h)dx = \phi(-h);$$

consequently $\delta(x+h)$ is the symbolic function for the distribution which assigns the value $\phi(-h)$ to every testing function $\phi(x)$.

Differentiation.

The derivative of a distribution T is defined in a manner similar to the definition for a function:

$$T'(\phi) = \lim_{h\to 0} \frac{\tau_h T(\phi) - T(\phi)}{h} \quad . \tag{2.11}$$

From (2.10) we get

$$T'(\phi) = \lim\left[T(\tau_{-h}\phi) - T(\phi)\right]/h = \lim T\left[\frac{\tau_{-h}\phi-\phi}{h}\right]$$

$$= \lim T\left[\frac{\phi(x-h)-\phi(x)}{h}\right] = T\left[\lim \frac{\phi(x-h)-\phi(x)}{h}\right]$$

$$= T(-\phi'(x)) = -T(\phi') \ ,$$

where use has been made of the fact that T is a continuous functional. We state this result as

Theorem II. If T is a distribution and ϕ is a testing function, then

$$T'(\phi) = -T(\phi') \ . \tag{2.12}$$

Let $t(x)$ be the symbolic function associated with T and let $t'(x)$ be the symbolic function associated with T'; then (2.12) takes the form

$$\int t'(x)\phi(x)dx = -\int t(x)\phi'(x)dx, \tag{2.13}$$

which is exactly the result that would be obtained if $t(x)$ were a real function and integration by parts were used.

To illustrate the concept of differentiation, consider the Heaviside unit function $H(x)$. Since it is a summable function, it may be considered as defining a distribution. As such what is its derivative? By (2.12)

$$H'(\phi) = -H(\phi') = -\int_0^\infty \phi'(x)dx = \phi(0) \ ,$$

since $\phi(x)$ vanishes for sufficiently large x; therefore, we may write

$$H' = \delta \tag{2.14}$$

since H' assigns the value $\phi(0)$ to every testing function $\phi(x)$. Note that the derivative of $H(x)$ considered as a function is zero everywhere except at the origin, where

it is undefined, while the derivative of the distribution defined by $H(x)$ is the δ-functional.

What is the meaning of δ'? By (2.12)

$$\delta'(\phi) = -\delta(\phi') = -\phi'(0); \tag{2.15}$$

hence the δ'-functional assigns the value $-\phi'(0)$ to every testing function $\phi(x)$. In the same way, we can prove that

$$\delta^{(k)}(\phi) = (-)^k\phi^{(k)}(0). \tag{2.16}$$

Suppose T is defined by an absolutely continuous function $f(x)$, that is, a function which is the integral of its derivative; then by integration by parts

$$-\int f(x)\phi'(x)dx = \int f'(x)\phi(x)dx;$$

consequently we find that the symbolic function for the derivative of the distribution defined by $f(x)$ is the same as the derivative of $f(x)$ considered as a function. However, if $f(x)$ is not absolutely continuous, the two derivatives are different. Henceforth, the derivative of $f(x)$, written $f'(x)$, will mean the symbolic function for the derivative of the distribution defined by $f(x)$.

We shall prove

Theorem III. Suppose $f(x)$ is absolutely continuous and differentiable except at the points $a_1, a_2 \cdots, a_n$, where $f(x)$ has jumps of magnitude $\alpha_1, \alpha_2 \cdots, \alpha_n$ respectively; then

$$f'(x) = [f'(x)] + \alpha_1\delta(x-a_1) + \alpha_2\delta(x-a_2) + \cdots + \alpha_n\delta(x-a_n)$$

where $[f'(x)]$ is the derivative of $f(x)$ considered as a function.

To prove this theorem, put

$$g(x) = f(x)-\alpha_1 H(x-a_1) - \alpha_2H(x-a_2) - \cdots -\alpha_nH(x-a_n). \tag{2.17}$$

It is clear that $g(x)$ will be absolutely continuous and differentiable, so that the derivative of $g(x)$ will be the same whether $g(x)$ is considered a distribution or a function. When (2.17) is differentiated as an equation involving distributions, we find that

$$f'(x) = g'(x) + \alpha_1\delta(x-a_1) + \alpha_2\delta(x-a_2) + \cdots + \alpha_n\delta(x-a_n). \tag{2.18}$$

Since $g'(x)$ is the same as the derivative of $f(x)$ considered as a function, we see that the derivative of a function considered as defining a distribution is the usual derivative plus the sum of δ-functions at the jumps, multiplied by the magnitude of the jumps.

In a similar way we see that

$$(2.19)\qquad f''(x) = \varphi''(x) + \alpha_1\delta'(x-a_1) + \cdots + \alpha_n\delta'(x-a_n)$$

$$+ \beta_1\delta(x-b_1) + \cdots + \beta_k\delta(x-b_k),$$

where $b_1, b_2, \cdots, b_k$ are the points of discontinuity in $\varphi'(x)$ and $\beta_1, \beta_2, \cdots, \beta_k$ are the corresponding jumps in the value of $\varphi'(x)$.

Examples of Derivatives of Distributions

Consider the distribution defined by the function $|x|$. Its derivative is the function which is -1 for the x negative and +1 for x positive. We call this function signum x and denote it by sgn x. We have

$$(2.20)\qquad \frac{d}{dx}\ \operatorname{sgn} x = 2\delta(x)\ ,$$

since the sgn function is constant except for a jump of magnitude 2 at the origin.

As an application of this result, consider

$$I = \int_{-1}^{1} |x|\psi''(x)dx\ .$$

Using repeated integration by parts, we find that

$$I = |x|\psi'(x)\Big|_{-1}^{1} - \int_{-1}^{1} |x|'\psi'(x)dx$$

$$= \Big[\,|x|\ \psi'(x) - |x|'\psi(x)\Big]_{-1}^{1} + \int_{-1}^{1} |x|''\psi(x)dx$$

$$= \psi'(1) - \psi'(-1) - \psi(1) - \psi(-1) + 2\psi(0)\ .$$

This method is simpler than the usual one where I is written as

$$-\int_{-1}^{0} x\psi''(x)dx + \int_{0}^{1} x\psi''(x)dx$$

and integration by parts is applied to both integrals.

As another illustration, consider the problem of maximizing the integral

$$K = \int_{-1}^{1} f(x)(x-x_1)(x-x_2)dx,$$

where $f(x)$ is an integrable real function such that $|f(x)| \le 1$, and where x_1, x_2 are arbitrary numbers between -1 and 1. For given x_1, x_2, K will be a maximum if $f(x) = +1$ when $(x-x_1)(x-x_2)$ is positive and if $f(x) = -1$ when $(x-x_1)(x-x_2)$ is negative; therefore

$$K \leq K^* = \int_{-1}^{1} (x-x_1)(x-x_2)\operatorname{sgn}\left[(x-x_1)(x-x_2)\right]dx \ .$$

Now K^* will be a maximum for those values of x_1 and x_2 which make

$$\frac{\partial K^*}{\partial x_1} = \frac{\partial K^*}{\partial x_2} = 0 \ .$$

We have

$$\frac{\partial K^*}{\partial x_1} = -\int_{-1}^{1}(x-x_2)\operatorname{sgn}\left[(x-x_1)(x-x_2)\right]dx$$

$$+\int_{-1}^{1} (x-x_1)(x-x_2)2\delta(x-x_1)dx \ .$$

The second integral is zero, since $(x-x_1)(x-x_2)$ vanishes when $x=x_1$. Finally,

$$\frac{\partial K^*}{\partial x_1} = -\int_{-1}^{x_1} (x-x_2)dx + \int_{x_1}^{x_2}(x-x_2)dx - \int_{x_2}^{1}(x-x_2)dx.$$

There is a similar formula for $\frac{\partial K^*}{\partial x_2}$. The remainder of the solution may be left to the reader.

Problems

2.3 If $f(x)$ is absolutely continuous, show that the derivative of the distribution defined by $f(x)H(x)$ is $f'(x)H(x) + f(0)\delta(x)$.

2.4 If $f(x)$ is absolutely continuous with zeroes at $x_1,\ldots,x_n$, the derivative of $\operatorname{sgn}(f(x))$ is

$$2\delta(x-x_1)\operatorname{sgn}f'(x_1) + \ldots + 2\delta(x-x_n)\operatorname{sgn}f'(x_n).$$

2.5 Show that the integral

$$u(x) = \int |x-\zeta|\phi(\zeta)d\zeta$$

satisfies the differential equation

$$u'' = 2\phi(x).$$

Hint. Use (2.20)

2.6 Show that the integral

$$u(x) = \int e^{ik|x-\zeta|}\phi(\zeta)d\zeta$$

satisfies the equation

$$u'' + k^2u = 2\phi(x).$$

Hint. Use (2.20) and the fact that $(\operatorname{sgn}x)^2 = 1$.

2.7 If $p(x)$ is an integrable non-negative function, show that

$$u = H(x)\int^{x} \frac{d\zeta}{p(\zeta)}$$

is a solution of

$$(pu')' = \delta(x).$$

Functions with Infinite Jumps

What is the derivative of the distribution corresponding to the function which is zero for negative x and $x^{-1/2}$ for positive x? The usual derivative is zero for negative x and $-x^{-3/2}/2$ for positive x. However, the latter function is not integrable and cannot define a distribution.

We should like to use formula (2.12) to define the derivative, but integration by parts cannot be applied to the integral

$$-\int_0^\infty \phi'(x)x^{-1/2}dx \ .$$

The solution is to realize that this integral is the limit as $\epsilon \to 0$ of the following integral:

$$-\int_\epsilon^\infty \phi'(x)x^{-1/2}dx.$$

Now integration by parts is possible and we have

$$\begin{aligned} -\int_0^\infty \phi'(x)x^{-1/2}dx &= -\lim_{\epsilon \to 0} \int_\epsilon^\infty \phi'(x)x^{-1/2}dx \\ &= \lim_{\epsilon \to 0} \left[\phi(\epsilon)\epsilon^{-1/2} - \frac{1}{2}\int_\epsilon^\infty \phi(x)x^{-3/2}dx\right] . \end{aligned} \tag{2.21}$$

This last expression was first considered by Hadamard. He called it the finite part of the integral

$$-\frac{1}{2}\int_0^\infty \phi(x)x^{-3/2}dx \ .$$

We generalize this concept to a function $g(x)$ which is not integrable because of a discontinuity at x=a. Consider the integral

$$\int_{a+\epsilon}^b g(x)dx \ .$$

This integral may have an asymptotic expansion in positive and negative powers of ϵ. The terms in negative powers of ϵ or in $\log \epsilon$ are called the infinite part of the integral. The limit of the difference between the integral and its infinite part as ϵ approaches zero is called the finite part of the integral. This finite part of the integral will be used to define distributions corresponding to non-integrable functions.

An important set of distributions is obtained in this way when we take $g(x) = x^p$ where the real part of $p \leq -1$. Assume first that p is not a negative integer and consider the finite part of the integral

$$\int_0^\infty x^p\phi(x)dx.$$

Suppose that m is the smallest integer such that the real part of $(p+m) > -1$; then by using the Taylor series for $\phi(x)$ it can be shown that the following integral exists

in the ordinary sense:

$$I_+(\phi) = \int_0^\infty x^p \left[\phi(x)-\phi(0)-x\phi'(0)-\frac{x^2}{2!}\phi''(0)- \ldots - \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right]dx$$

$$(2.22) \quad = \lim_{\epsilon \to 0} \int_\epsilon^\infty x^p \left[\phi(x)-\phi(0)-x\phi'(0)- \ldots - \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right] dx$$

$$= \lim_{\epsilon \to 0} \left[\int_\epsilon^\infty x^p\phi(x)dx + \frac{\phi(0)\epsilon^{p+1}}{p+1} + \frac{\phi'(0)\epsilon^{p+2}}{p+2} + \frac{\phi''(0)}{2!}\frac{\epsilon^{p+3}}{p+3} + \ldots + \frac{\phi^{(m-1)}(0)}{(m-1)!}\frac{\epsilon^{p+m}}{p+m}\right].$$

This shows that $I_+(\phi)$ is the finite part of the integral

$$\int_0^\infty x^p\phi(x)dx.$$

If x goes from $-\infty$ to 0, let us consider the finite part of the integral

$$\int_{-\infty}^0 x^p\phi(x)dx.$$

It will be

$$(2.23) \qquad I_-(\phi) = \int_{-\infty}^0 x^p\left[\phi(x)-\phi(0)- \ldots - \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right] dx.$$

We shall use the symbol $('x)^p$ to represent the symbolic function for the distributions whose values are given by $I_+(\phi)+I_-(\phi)$; that is,

$$(2.24) \qquad \int ('x)^p\phi(x)dx = I_+(\phi)+I_-(\phi).$$

The symbolic function for $I_+(\phi)$ will be denoted by $('x)^pH(x)$; consequently

$$(2.25) \qquad \int ('x)^pH(x)\phi(x)dx = \int_0^\infty ('x)^p\phi(x)dx = I_+(\phi).$$

Similarly, $('x)^pH(-x)$ will denote the symbolic function corresponding to $I_-(\phi)$.

If p is a negative integer, say $p = -m$, then the finite part of the integral

$$\int_0^\infty x^p\phi(x)dx$$

will be

$$\lim_{\epsilon\to 0}\left[\int_\epsilon^\infty x^{-m}\phi(x)dx-\phi(0)\frac{\epsilon^{-m+1}}{m-1} - \phi(0)\frac{\epsilon^{-m+2}}{m-2} - \ldots - \frac{\phi^{(m-1)}(0)}{(m-1)!}\log\epsilon\right]$$

$$= \lim_{\epsilon\to 0}\int_\epsilon \frac{1}{x^m}\left[\phi(x)-\phi(0)-x\phi'(0)-\ldots-\frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right] dx$$

$$(2.26)$$

$$+\int_1^\infty x^{-m}\left[\phi(x)-\phi(0)-x\phi'(0)-\ldots-\frac{x^{m-2}}{(m-2)!}\,\phi^{(m-2)}(0)\right]dx$$

$$=\int_0^1 x^{-m}\left[\phi(x)-\phi(0)-x\phi'(0)-\ldots-\frac{x^{m-1}}{(m-1)!}\,\phi^{(m-1)}(0)\right]dx$$

$$+\int_1^\infty x^{-m}\left[\phi(x)-\phi(0)-x\phi'(0)-\ldots-\frac{x^{m-2}}{(m-2)!}\,\phi^{(m-2)}(0)\right]dx\ .$$

As an application of these ideas consider the derivative of the distribution defined by the integrable function $\log|x|$. Using Theorem III, we have

$$\begin{aligned}-\int \log|x|\phi'(x)dx &= -\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}+\int_\epsilon^\infty\right]\log|x|\phi'(x)dx\\ &= \lim_{\epsilon\to 0}\left[\log\epsilon\Big(\phi(\epsilon)-\phi(-\epsilon)\Big)+\left(\int_{-\infty}^{-\epsilon}+\int_\epsilon^\infty\right)\frac{\phi(x)}{x}\,dx\right]\\ &= \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}+\int_\epsilon^\infty\right]\frac{\phi(x)}{x}\,dx\ ,\end{aligned}\tag{2.27}$$

because since the testing function $\phi(x)$ is continuous and differentiable at the origin, the term arising from the integration by parts is zero. The last expression in (2.27) is customarily called the <u>principal value</u> of the integral and is represented by the symbol

$$\text{P.V.}\int\frac{\phi(x)}{x}\,dx\ .$$

We shall use $('x)^{-1}$ to represent the symbolic function for the distribution whose values are given by this expression; then

$$\int\frac{\phi(x)}{'x}\,dx=\text{P.V.}\int\frac{\phi(x)}{x}\,dx=\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}+\int_\epsilon^\infty\right]\frac{\phi(x)}{x}\,dx.\tag{2.28}$$

Equation (2.27) now shows that

$$(\log|x|)'=('x)^{-1}\ .\tag{2.29}$$

By the use of (2.29) we may find the derivative of $\log x$. But first, $\log x$ must be defined for all values of x from $-\infty$ to $+\infty$. For x positive, $\log x=\log|x|$, while for x negative we assume $\log x=\log|x|+i\pi$. We may then write

$$\log x=\log|x|+i\pi\,H(-x)\tag{2.30}$$

and then

$$(\log x)'=\frac{1}{'x}-i\pi\delta(x)\ ,\tag{2.31}$$

a formula already given by Dirac.

<u>Problems</u>

2.8 If $\phi(x)$ is a testing function prove that

$$\int \frac{\phi(x)dx}{{}'x} = \lim_{\eta \to 0} \int \frac{\eta\phi(x)dx}{x^2 + \eta^2} \ .$$

Problem 2.9 If $a > 0$, prove that

$$\int_0^\infty ({}'x)^{p-1} e^{-ax} dx = \frac{\Gamma(p)}{a^p}$$

for all values of p, except p = 0, -1, -2,...

Problem 2.10 Prove that

$$\frac{d}{dx}\left[({}'x)^p H(x)\right] = p({}'x)^{p-1} H(x)$$

if p is not a negative integer.

Problem 2.11. If m is an integer prove that

$$\frac{d}{dx}\left[({}'x)^{-m} H(x)\right] = -m({}'x)^{m-1} H(x) + (-)^m \frac{\delta^{(m)}}{m!} \ .$$

Problem 2.12. If m is an integer, prove that

$$\frac{d}{dx} ({}'x)^{-m} = -m({}'x)^{-m-1} \ .$$

Multiplication of Distributions

In general, two arbitrary distributions cannot be multiplied. For example, consider the distribution associated with the integrable function f(x). If $f(x)^2$ is not integrable, then the product of the distribution with itself is not a distribution.

One case in which we are certain the product will always exist as a distribution is when one of the factors is defined by an analytic function. Let $\psi(x)$ be an analytic function. We define ψT as follows:

$$\psi T(\phi) = T(\psi\phi). \tag{2.32}$$

Note that if ψ is analytic and ϕ is a testing function then $\psi\phi$ is also a testing function and so the right side of (2.32) always has a meaning. The reasons for the definition (2.32) can be seen if we assume T is defined by a function f(x); then

$$\psi f(\phi) = \int \psi(x) f(x) \phi(x) dx = \int f(x) \psi(x) \phi(x) dx = f(\psi\phi) \ ,$$

which is the same as (2.32).

We shall illustrate the definition by some examples. Consider $x \cdot \frac{1}{{}'x}$. By (2.32)

$$x \cdot \frac{1}{{}'x}(\phi) = \frac{1}{{}'x}(x\phi) = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right] \phi(x)dx = \int \phi(x)dx = 1(\phi);$$

consequently

$$x \cdot \frac{1}{{}'x} = 1.$$

As another example, consider $x\delta(x)$. By (2.32)

$$x\delta(\phi) = \delta(x\phi) = 0,$$

since the value of $x\phi(x)$ at the origin is zero. We may write

(2.33) $$x\delta(x) = 0.$$

Even though the product was defined only when one factor is an analytic function, it is possible in some cases to extend the definition. Suppose $f(x)$ is a function continuous at the origin; then we may define $f(x)\delta(x)$ by using (2.32):

$$f\delta(\phi) = \delta(f\phi) = f(0)\phi(0);$$

end we see that

(2.34) $$f(x)\delta(x) = f(0)\delta(x).$$

The derivative of a product ψT will follow from (2.32) and from the definition of derivative. We have

$$(\psi T)'(\phi) = -\psi T(\phi') = -T(\psi\phi') = -T[(\psi\phi)' - \psi'\phi]$$

$$= -T[(\psi\phi)'] + T(\psi'\phi) = T'(\psi\phi) + T(\psi'\phi)$$

$$= (\psi T' + \psi' T)(\phi),$$

and thus we obtain the usual theorem for differentiating a product:

Theorem IV. If T is a distribution and ψ a testing function, then $(\psi T)' = \psi T' + \psi' T$.

If this is applied to (2.33) we get

$$0 = [x\delta(x)]' = \delta(x) + x\delta'(x)$$

or

$$x\delta'(x) = -\delta(x) .$$

The n-th derivative of (2.33) gives the general result:

(2.35) $$x\delta^{(n)}(x) = -n\delta^{(n-1)}(x) .$$

It should be noted that in certain cases multiplication may not be associative. For example by (2.33)

$$\frac{1}{'x} \cdot (x\delta) = 0$$

while

$$\left(\frac{1}{'x}x\right) \cdot \delta = \delta.$$

Problems

2.13 If $q \geq 0$ prove that

$$x^q('x)^p = ('x)^{p+q} .$$

2.14 If n and m are positive integers prove that

a) $x^m \delta^{(m)}(x) = (-)^m m!\, \delta(x)$;

b) $x^{m+n}\delta^{(m)}(x) = 0$;

c) $x^m\delta^{(m+n)}(x) = (-)^m \frac{(m+n)!}{n!}\, \delta^{(n)}(x)$.

Hint. Use (2.30). Note $\delta^{(0)}(x)$ is $\delta(x)$, $\delta^{(1)}(x)$ is $\delta'(x)$, etc.

Problems

2.15 If $g(x)$ is a function which is continuously differentiable at the origin, show with the help of (2.30) that

$$g(x)\delta'(x) = -g'(0)\delta(x) + g(0)\delta'(x).$$

Compare this result with that obtained by differentiating both sides of the identity

$$g(x)\delta(x) = g(0)\delta(x).$$

2.16 Show that the results of Problem 2.10 and 2.11 may be obtained formally by differentiating $('x)^{-m}H(x)$ as a product and using Problem 2.14a.

Convergence of Distributions

A sequence of distributions T_k is said to converge to zero in the weak sense if the numbers $T_k(\phi)$ converge to zero for every testing function $\phi(x)$. The sequence T_k converges to zero in the strong sense if the numbers $T_k(\phi)$ converge to zero for all testing functions $\phi(x)$ and if the convergence is uniform for all testing functions which are zero outside the same fixed interval and have their derivatives uniformly bounded. It can be shown that the two definitions are equivalent*. We shall usually work with the first definition.

The sequence of distributions T_k will be said to converge to the distribution T if the sequence of distributions $T_k - T$ converges to zero. If we express the definition of the convergence of distributions in terms of the corresponding symbolic functions, we have the following useful result:

Theorem V. If $t_k(x)$ is a sequence of symbolic functions which converges to $t(x)$, and if $\phi(x)$ is any testing function, then

$$\lim \int t_k(x)\phi(x)dx = \int t(x)\phi(x)dx.$$

Of course, the convergence is that for distributions, so this theorem is really just a re-statement of the definition of convergence. Suppose, however, that the symbolic functions $t_k(x)$ were actual functions, as they would be if the corresponding distributions were defined by integrable functions: How would the limit of $t_k(x)$ in the ordinary sense, compare with the limit in the sense of distributions? There are

---

* L. Schwartz, Theorie des Distributions , op. cit., tome 1, p.73.

many possibilities: first, the limit in the sense of ordinary point-wise convergence of $t_\kappa(x)$ may not exist while the distribution limit may exist, second, the two limits may exist but be different; third, the limits may exist and be the same.

As an illustration of the first possibility consider the limit as $a \to 0$ of the set of functions

$$t(x,a) = \frac{a}{\pi(x^2+a^2)} .$$

For $x \neq 0$, the limit of $t(x,a)$ in the ordinary sense is zero, while for $x = 0$ the limit in the ordinary sense does not exist. We shall prove that in the sense of distributions,

$$\lim_{a\to 0} \frac{a}{\pi(x^2+a^2)} = \delta(x). \tag{2.36}$$

Let $\phi(x)$ be a testing function. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{a\phi(x)dx}{\pi(x^2+a^2)} = \int \frac{a[\phi(x)-\phi(0)]}{\pi(x^2+a^2)} dx + \frac{\phi(0)}{\pi} \int \frac{a\,dx}{x^2+a^2}$$

$$= \left[\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}\right] \frac{a[\phi(x)-\phi(0)]}{\pi(x^2+a^2)} dx + \int_{-\eta}^{\eta} \frac{a[\phi(x)-\phi(0)]}{\pi(x^2+a^2)} dx + \phi(0) \tag{2.37}$$

for any value of η .

Since $\phi(x)$ is continuous at $x = 0$, there exists a number η such that

$$|\phi(x) - \phi(0)| < \epsilon$$

for any arbitrary value of $\epsilon > 0$ and for $-\eta \leq x \leq \eta$; therefore, the second integral on the right of (2.37) can be estimated as follows:

$$\left| \int_{-\eta}^{\eta} \frac{a|\phi(x)-\phi(0)|}{\pi(x^2+a^2)} dx \right| \leq \frac{a\epsilon}{\pi} \int_{-\eta}^{\eta} \frac{dx}{x^2+a^2} < \frac{a\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \epsilon.$$

Since $\phi(x)$ is a testing function, it is bounded for all x. Let M be a bound for $\phi(x)$; then

$$|\phi(x) - \phi(0)| < 2M$$

for all x, and the first integral on the right of (2.37) is less in absolute value than

$$\left[\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}\right] \frac{2Ma}{\pi(x^2+a^2)} dx = \frac{4M}{\pi} \left(\frac{\pi}{2} - \arctan \frac{\eta}{a}\right)$$

$$= \frac{4M}{\pi} \operatorname{arccot} \frac{\eta}{a} = \frac{4M}{\pi} \arctan \frac{a}{\eta} .$$

Once the value of η is determined, a can be chosen small enough so that

$$\frac{4M}{\pi} \arctan \frac{a}{\eta} < \epsilon .$$

Consequently,

$$|I-\phi(0)| < 2\epsilon$$

for all values of a which are sufficiently near to zero. Since ϵ can be made arbitrarily small, it follws that

$$\lim_{a\to 0} I = \lim_{\epsilon\to 0} \int \frac{a\phi(x)dx}{\pi(x^2+a^2)} = \phi(0) .$$

This result proves (2.36).

The second possibility, that the limit in the ordinary sense may exist but be different from the limit in the sense of distributions, is illustrated by the derivatives of the set of functions given above, i.e.,

$$t(x,a)' = -\frac{2ax}{\pi(x^2+a^2)^2} .$$

For all values of x the limit of $t(x,a)'$ in the ordinary sense is zero, but we shall prove that the limit in the sense of distributions is $\delta'(x)$.

The effect of $t(x,a)'$ on any testing function is found as follows:

$$J = -\int \frac{2ax\phi(x)dx}{\pi(x^2+a^2)^2} = -\int \frac{2ax[\phi(x)-\phi(0)]}{\pi(x^2+a^2)^2}\, dx - \phi(0)\int \frac{2ax}{\pi(x^2+a^2)^2}\, dx$$

$$= \frac{a[\phi(x)-\phi(0)]}{\pi(x^2+a^2)}\Bigg|_{-\infty}^{\infty} \int \frac{a\phi'(x)}{\pi(x^2+a^2)}\, dx = -I(\phi') .$$

In the computation we have used the fact that

$$\int \frac{xdx}{(x^2+a^2)^2} = 0.$$

Since $\lim_{a\to 0} I(\phi') = \phi'(0)$, we have

$$\lim_{a\to 0} J(\phi) = -\lim_{a\to 0} I(\phi') = -\phi'(0),$$

and consequently we have proved that

$$\lim_{a\to 0} t(x,a)' = \delta'(x).$$

The third possibility, that the limits in both senses are the same, will occur in the case stated in

Theorem VI. If a sequence of integrable functions $t_k(x)$ converges in the ordinary sense to a function $t(x)$, and if the sequence $t_k(x)$ is bounded by an integrable function $f(x)$, i.e,

$$|t_k(x)| < f(x), \quad k = 1, 2, \ldots$$

then the limit of the sequence of distributions defined by $t_k(x)$ is the distribution defined by $t(x)$.

Consider the distributions defined by $t_k(x)$. These are the integrals

$$T_k(\phi) = \int t_k(x)\phi(x)dx.$$

We now use a well-known theorem on Lebesgue integration* which states that under the hypothesis of Theorem VI

$$\lim \int t_k(x)\phi(x)dx = \int t(x)\phi(x)dx.$$

This result implies that

$$\lim T_k(\phi) = T(\phi)$$

and so proves the theorem.

In the discussion of the second possibility we proved that

$$\lim_{a\to 0} \frac{d}{dx} t(x,a) = \frac{d}{dx} \lim_{a\to 0} t(x,a).$$

This result is no coincidence. We shall see that it follows from

Theorem VII. If a sequence of distributions T_k converge to a limit distribution T, then the sequence of derivatives T_k' converges to T', the derivative of the limit.

By the definition of the derivative (2.12),

$$T_k'(\phi) - T'(\phi) = T(\phi') - T_k(\phi').$$

Since T_k converging to T means that $T_k(\phi) - T(\phi)$ converges to zero for any testing function $\phi(x)$, and since $\phi'(x)$ is a testing function whenever $\phi(x)$ is, it follows that $T_k'(\phi) - T'(\phi)$ converges to zero for any testing function $\phi(x)$. This implies that T_k' converges to T' and so proves the theorem.

Since the limit of the derivatives converges to the derivative of the limit, we may apply Theorem VII to the derivatives and conclude that the limit of the second derivatives is the second derivative of the limit. Of course this process may be repeated. We have, therefore, the following

Theorem VIII. If a sequence of distributions T_k converges to a limit distribution T, then the sequence of n-th derivatives $T_k^{(n)}$ converges to $T^{(n)}$, the n-th derivative of the limit.

This result can also be applied to convergent series of distribution. A series of distributions

$$S_1 + S_2 + \ldots$$

will be said to converge if the partial sum

$$T_k = S_1 + S_2 + \ldots + S_k$$

---

* Titchmarsh, Theory of Functions, page 345. The Clarendon Press, Oxford, 1932

converges in the sense of distributions. We may now re-state Theorem VIII so that it applies to series:

Theorem IX. If a series of distributions

$$S_1 + S_2 + \dots$$

converges in the sense of distributions to a limit T, then the series of n-th derivatives

$$S_1^{(n)} + S_2^{(n)} + \dots$$

converges to $T^{(n)}$.

As an illustration of these theorems, consider the function $\log(x+ia)$ where a is a non-negative number. We have by (2.30)

$$\lim_{a\to 0} \log(x+ia) = \log|x| + i\pi H(-x)$$

and then by (2.31) and Theorem VII

$$\lim_{a\to 0} \frac{1}{x+ia} = \lim_{a\to 0} \left[\frac{x}{x^2+a^2} - i \frac{a}{x^2+a^2} \right] = \frac{1}{'x} - i\pi\delta(x).$$

This formula proves again that

$$\lim_{a\to 0} \frac{a}{\pi(x^2+a^2)} = \delta(x)$$

and proves also that

$$\lim_{a\to 0} \frac{x}{x^2+a^2} = \frac{1}{'x} ,$$

a result which we have obtained in Problem 2.8.

The theorems of this section show that when we work with distributions or symbolic functions we may without danger interchange the operation of taking the limit with the operations of integration or differentiation. We now prove a theorem which shows that we may differentiate under the integral sign.

Theorem X. If $\phi(x,\lambda)$ is a testing function of x for all values of λ in a neighborhood of λ_0, and if $\frac{\partial\phi}{\partial\lambda}$ is a testing function for $\lambda = \lambda_0$, then

$$\frac{d}{d\lambda} T(\phi)\Big|_{\lambda=\lambda_0} = T\left(\frac{\partial\phi}{\partial\lambda}\Big|_{\lambda=\lambda_0} \right) ,$$

or expressed in terms of symbolic functions,

$$\frac{d}{d\lambda} \int t(x)\phi(x,\lambda)dx = \int t(x) \frac{\partial\phi}{\partial\lambda}\Big|_{\lambda_0} dx .$$

From the definition of a derivative, we have

$$\frac{d}{d\lambda} T(\phi)\Big|_{\lambda=\lambda_o} = \lim_{\Delta\lambda\to 0} \frac{T[\phi(x,\lambda_o+\Delta\lambda)] - T[\phi(x,\lambda_o)]}{\Delta\lambda}$$

$$= \lim T\left[\frac{\phi(x,\lambda_o+\Delta\lambda)-\phi(x,\lambda_o)}{\Delta\lambda}\right] .$$

Now, because a distribution is a continuous function, or, what is equivalent, because of Theorem I, we may take the limit inside the distribution and get

$$\frac{d}{d\lambda} T(\phi)\Big|_{\lambda=\lambda_o} = T\left[\lim \frac{\phi(x,\lambda_o+\Delta\lambda)-\phi(x,\lambda_o)}{\Delta\lambda}\right]$$

$$= T\left(\frac{\partial\phi}{\partial\lambda}\right)\Big|_{\lambda=\lambda_o} .$$

This completes the proof of the theorem.

We shall apply Theorem X to get a result which will be useful when we apply the theory of distributions to the solution of differential equations. We start with a simple result from the theory of functions of a complex variable. Consider the contour integral

$$I_R = \int_S \frac{e^{ikx}}{k}\, dk$$

where S is the positively described semi-circle of radius R in the upper half plane. We shall use Theorem VI to obtain the limit in the sense of distributions of I_R as R goes to infinity. First, suppose $x > 0$ and put $k = Re^{i\theta}$; then

$$I_R = i\int_o^\pi \exp(ixRe^{i\theta})d\theta .$$

Now,

$$|\exp(ixRe^{i\theta})| \le e^{-Rx\sin\theta};$$

therefore we see that

$$|I_R| \le 2\int_o^{\pi/2} e^{-Rx\sin\theta}\, d\theta.$$

It is easy to show (see Problem 2.18) that for $0 \le \theta \le \pi/2$,

$$\sin\theta \ge 2\theta/\pi;$$

consequently we have

$$|I_R| \le 2\int_o^{\pi/2} e^{-2Rx\theta/\pi} d\theta = \frac{\pi}{Rx}(1-e^{-Rx}),$$

and this shows that for $x > 0$ the limit in the ordinary sense of I_R is zero. Second, for $x = 0$, again put $k = Re^{i\theta}$; then

$$I_R = \pi i.$$

Finally, for $x < 0$ the contour may be replaced by the semi-circle S' of radius R in the lower half-plane if we take into account the residue at the origin. We have then

$$I_R = -\int_{S'} \frac{e^{ikx}}{k}\, dk + \text{residue}.$$

The integral over S' approaches zero for $x < 0$ in the same way that the integral over S approaches zero for $x > 0$. Since the residue is $2\pi i$, we have finally

$$\lim I_R = 2\pi i.$$

Clearly, the integral I_R converges boundedly to its limit; therefore from Theorem VI we know that the limit in the sense of distributions of I_R is $2\pi i H(-x)$.

We wish to investigate what is the result of differentiating the limit of I_R with respect to x. From Theorem VII we have

$$-2\pi i\, \delta(x) = \frac{d}{dx} \lim I_R = \lim \frac{d}{dx} I_R ,$$

and from Theorem X we have

$$\frac{d}{dx} I_R = \int_S ie^{ikx} dk;$$

consequently, we get the following useful result:

$$\lim \int_S e^{ikx} dk = -2\pi\delta(x). \tag{2.38}$$

Differentiating (2.38) repeatedly with respect to x, we get

$$\lim \int_S (ik)^n e^{ikx} dk = -2\pi\delta^{(n+1)}(x). \tag{2.39}$$

Here $\delta^{(n+1)}(x)$ means the (n+1)-th derivative of $\delta(x)$.

By Cauchy's Theorem, the integral over S is the same as the integral from -R to R along the real axis; therefore (2.39) becomes

$$\lim_{R\to\infty} \int_{-R}^{R} (ik)^n e^{ikx} dk = 2\pi\delta^{(n+1)}(x). \tag{2.40}$$

Problems

2.17 Prove that the limit in the sense of distributions of e^{ikx} as k approaches infinity is zero. Prove also that

$$\lim_{k\to\infty} k^n e^{ikx} = 0$$

for n = 1,2,3,... Hint. Consider $\int \phi(x) e^{ikx} dx$ and use integration by parts.

2.18 Prove the statement in the text that for $0 \le \theta \le \pi/2$,

$$\sin\theta \ge \frac{2}{\pi}\,\theta.$$

Hint. Consider the graph of $y = \sin x$ and $y = \frac{2}{\pi} x$.

2.19 Prove that

$$\lim_{R\to\infty} \int_0^R e^{ikx}dk = \lim_{\epsilon\to 0} \int_0^\infty e^{-\epsilon k} e^{ikx}dk = \pi\delta(x) + \frac{i}{x}$$

2.20 Use Problem 2.19 to evaluate

$$\lim_{R\to\infty} \int_0^R (ik)^n e^{ikx}dk \ .$$

Values of Distributions and Change of Variables

It has been stated previously that in contrast to the case of functions, a distribution does not have values for different values of x. However, for some purposes it is convenient to define the values of a distribution. First, we define the support of a testing function $\phi(x)$ as the closure of the set of points over which $\phi(x) \neq 0$. We now define the sense in which a distribution has the value zero.

A distribution T is zero in an open set* Ω of $T(\phi) = 0$ for every testing function ϕ whose support is contained in Ω. Two distributions T_1 and T_2 are said to be equal in Ω if $T_1 - T_2$ is zero in Ω. Notice that these definitions do not give the value of the distribution at a point but only in the neighborhood of a point. Also, notice that these values are found by using testing functions which are zero except for the neighborhood in which we are interested.

We define the support of a distribution as the closed set which is the complement of the largest open set in which T = 0. If F is the support of T then it is clear that the value of $T(\phi)$ will depend only on the values that ϕ takes in F. This permits us to extend the class of testing functions when T has a bounded support, for then, since $T(\phi)$ will not depend on the values of ϕ for large x, we need no longer require that a testing function vanish outside a finite interval. In the language of symbolic functions we may express this fact by saying that if T has a bounded support, then t(x) vanishes outside a finite interval and so

$$T(\phi) = \int t(x)\phi(x)dx$$

is really an integral over a finite interval.

Let us investigate the values of $\delta(x)$. In the neighborhood of any point not the origin its value is zero. In the neighborhood of the origin the value of $\delta(x)$ is not defined; consequently, the support of $\beta(x)$ is the origin. Since the support is a bounded set, the δ-functional may be applied to any continuous function without any restriction as to behaviour for large values of x. Similar conclusions apply to any derivative of the δ-functional.

---

* An open set Ω is a set of points such that if P belongs to Ω then all the points of some interval containing P belong to Ω.

We now wish to define the meaning of a change of variable in a distribution or in a symbolic function. Let $f(x)$ be an absolutely continuous monotonic function of x and such that $f'(x) \neq 0$; then if $t(x)$ is an integrable function and $\phi(x)$ a testing function we have

$$\int t(f(x))\phi(x)dx = \int t(y)\phi(x) \frac{dy}{|f'(x)|} \tag{2.41}$$

by making the change of variable $y = f(x)$. The absolute value of the derivative is needed in order to ensure that the range of integration should always be from $-\infty$ to $+\infty$.

We use (2.41) to define $t(f(x))$ in case $t(x)$ is not an integrable function. For example, $\delta(x^3+3x-4)$ is defined by

$$\int \delta(x^3+3x-4)\phi(x)dx = \int \delta(y) \frac{\phi(x)}{3(x^2+1)} dy = \frac{\phi(1)}{6}$$

since $x = 1$ when $y = 0$. We then have

$$\delta(x^3+3x-4) = \frac{\delta(x-1)}{6} .$$

In a similar way we can show that

$$\delta(f(x)) = \frac{\delta(x-x_o)}{|f'(x_o)|} \tag{2.42}$$

where x_o is the root, if any exists, of the equation

$$f(x) = 0.$$

We shall extend (2.42) to the case where $f(x)$ is not monotonic, i.e., a case when $f'(x)$ may equal zero. We put

$$\delta(f(x)) = \sum_k \frac{\delta(x-x_k)}{|f'(x_k)|} . \tag{2.43}$$

The numbers x_k are the zeros of the equation

$$f(x) = 0.$$

As an illustration of (2.42) we quote a formula given by Dirac:

$$\delta(x^2-a^2) = \frac{1}{2a} [\delta(x-a)+\delta(x+a)].$$

Finally, it is easy to see from (2.41) that

$$\frac{d}{dx} t(f(x)) = t'(f(x))f'(x) ,$$

which is the same as the usual formula for differentiating a function of a functions. It is to be noted that here the derivative is in the sense of distributions.

Problems

2.21 Suppose $f(x)$ is an absolutely continuous function; then $H(f(x))$ equals 1 when $f(x) > 0$ and equals zero when $f(x) < 0$. Use the definition of the derivative of a distribution to evaluate $\frac{d}{dx} H(f(x))$ and show that

$$\frac{d}{dx} H(f(x)) = f'(x)\delta(f(x)),$$

where $\delta(f(x))$ is defined by (2.42).

2.22. Start with the power series for $\log(1-Z)$ and show that

$$\sum_1^\infty \frac{Z^{n+1}}{n(n+1)} = (1-Z)\log(1-Z) + Z.$$

Prove that the series converges uniformly and boundedly for all Z on the unit cirlce.

2.23 Prove that

$$\sum_1^\infty \frac{e^{in\theta}}{n} = -\frac{1}{2}\log(2-2\cos\theta) + i\,\frac{\pi-(\theta-2\pi[\frac{\theta}{2\pi}])}{2}.$$

Here $[\frac{\theta}{2\pi}]$ means the largest integer in $\theta/2\pi$. (Hint. Put $Z = e^{i\theta}$ in Problem 2.22 and differentiate.)

2.24 Prove that in the sense of distributions,

$$\sum_1^\infty \frac{\cos\theta n}{n} = -\frac{1}{2}\log(2-2\cos\theta)$$

$$\sum_1^\infty \frac{\sin\theta n}{n} = \frac{\pi}{2} - \frac{\theta-2\pi[\frac{\theta}{2\pi}]}{2}$$

$$\sum_1^\infty \sin n\theta = -\frac{1}{2}\cot\frac{\theta}{2}$$

$$\frac{1}{2} + \sum_1^\infty \cos n\theta = \pi\sum_{-\infty}^{\infty}\delta(\theta-2\pi k).$$

(Hint. Use Problem 2.23 and Theorem IX.)

Integration

Suppose a distribution S is given and we wish to find a distribution T such that

(2.44) $$T' = S.$$

If this were a relation between functions, there would be an infinite number of solutions of (2.44) differing from each other by a constant. We shall show that the same is true for distributions or, what is equivalent, that the only solution of

(2.45) $$T' = 0$$

is T equal to a constant.

The proof follows very closely the proof in Chapter I that a bounded linear functional is a scalar product. Suppose T satisfies (2.45); then consider the null space of T, i.e., the set of all testing functions $\psi(x)$ such that $T(\psi) = 0$. From (2.45)

$$T'(\phi) = -T(\phi') = 0$$

for any testing function $\phi(x)$. This shows that the set of derivatives of testing functions belong to the null space of T. Now, when is ψ equal to some ϕ'? If it is equal, then

$$\int \psi \, dx = \int \phi' dx = 0,$$

since $\phi(x)$ vanishes outside a finite interval.

Suppose, now, that $\phi_0(x)$ is a testing function such that

$$\int \phi_0(x) dx = 1.$$

Put

$$\int \phi(x) dx = \alpha;$$

then $\phi(x) - \alpha\phi_0(x)$ is a testing function such that

$$\int [\phi(x) - \alpha\phi_0(x)] dx = 0;$$

consequently $\phi(x) - \alpha\phi_0(x) = \psi(x)$ is the derivative of some testing function and therefore belongs to the null space of T. We now have

$$0 = T(\psi) = T(\phi - \alpha\phi_0) = T(\phi) - \alpha T(\phi_0)$$

or

$$T(\phi) = \alpha T(\phi_0) = T(\phi_0) \int \phi dx .$$

This proves that T equals the constant $T(\phi_0)$.

Convolutions or Folding of Distributions

We have seen that in general the product of two distributions does not exist. However, a new type of product, called the convolution or the Faltung or the fold*,

*. This term is due to Professor Ewald.

can be defined. For integrable function $f(x)$ and $g(x)$, the fold $h(x)$ is given by the formula

(2.46) $h(x) = \int f(x-y)g(y)dy = \int f(y)g(x-y)dy.$

We write

$$h = f*g.$$

It is easy to show that this product is both commutative and associative.

To extend (2.46) to distributions, a functional definition must be found. Let $\phi(x)$ be any testing function; then

$$h(\phi) = \int \phi(x)dx \int f(x-y)g(y)dy = \iint \phi(x)f(x-y)g(y)dydx$$

(2.47) $$= \iint f(\xi)g(\eta)\phi(\xi+\eta)d\xi\, d\eta\,,$$

where we have put $\xi = x-y$, $\eta = y$. The last integral may be considered as a repeated functional. Start with $\phi(\xi+\eta)$. Take its g-functional with respect to η; the result will be a function of ξ, say $\psi(\xi)$. This function is infinitely often differentiable. If $g(x)$ has a bounded support, then $\psi(\xi)$ will vanish outside a finite interval and so it will be a testing function. Now, by (2.47)

$$h(\phi) = f(\psi).$$

If $g(x)$ does not have a bounded support, then $\psi(\xi)$ need not vanish outside a finite interval, so it will not be a testing function. However, $f(\psi)$ will still exist if $f(x)$ has a bounded support. We conclude then that $h(\phi)$ always exists if either $f(x)$ or $g(x)$ has a bounded support.

Let S and T be two arbitrary distributions, one of which has a bounded support. The fold of S and T is defined as follows:

(2.48) $$S*T(\phi) = S_\xi\left\{ T_\eta\left[\phi(\xi+\eta)\right]\right\}.$$

It is easy to show that the fold is both commutative and associative.

As an illustration of folding, consider $\delta*T$. By (2.48)

$$\delta*T(\phi) = \delta_\xi\left\{ T_\eta\left[\phi(\xi+\eta)\right]\right\} = T_\eta\left\{\delta_\xi\left[\phi(\xi+\eta)\right]\right\} = T_\eta\left[\phi(\eta)\right] = T(\phi)\ ;$$

therefore

(2.49) $$\delta*T = T.$$

with symbolic functions this would be written

$$t(x) = \int \delta(x-y)t(y)dy.$$

Similarly, the fold $\delta'*T$ can be evaluated. By (2.48)

$$\delta'*T(\phi) = T_\eta\left\{\delta'_\xi\left[\phi(\xi+\eta)\right]\right\} = -T_\eta\left[\phi'(\eta)\right] = T'(\phi)\ ;$$

therefore

$$\delta' * T = T'.$$

A similar argument shows that if D is any differential operator, then

(2.50) $$D\delta * T = DT.$$

Also, by (2.50) and the associative law,

(2.51) $$D(S*T) = D\delta*(S*T) = (D\delta*S)*T = DS*T = S*DT;$$

thus we see that to differentiate a fold, only one of the factors need be differentiated.

These relations are very useful in the theory of differential equations. The solution of the differential equation

(2.52) $$DT = R,$$

where R is a given distribution, can be found once the solution of the following equation:

(2.53) $$DS = \delta$$

is known. For if we put

$$T = S*R$$

then, by (2.51), (2.52), and (2.49) we find that

$$DT = DS*R = \delta*R = R.$$

The distribution S satisfying (2.53) will be called the fundamental solution of the differential equation (2.52).

Problems

2.25 Show that for $n > 2$, $S = -\frac{H(x)}{(n-2)x^{n-2}}$ is a fundamental solution of the equation

$$(x^{n-1}t')' = r.$$

For $n = 2$, $S = H(x)\log x$.

Distribution in More than One Dimension

All the previous ideas can be carried over to an n-dimensional space. A testing function ϕ is defined to be a function of the n variables $x_1, x_2, \ldots, x_n$ which is infinitely differentiable and which vanishes outside some bounded region. A distribution is a linear continuous functional on the space of testing functions. Any function $f(x_1, x_2, \ldots, x_n)$ which is summable in n dimensions defines a distribution by the following formula:

$$f(\phi) = \int \ldots \int f(x_1, \ldots, x_n)\, \phi(x_1, \ldots, x_n)\, dx_1, \ldots, dx_n.$$

The derivative of a distribution T is defined by the following formula which is similar to that used in the case of distributions of one variable:

$$\frac{\partial T}{\partial x_k}(\phi) = -T\left(\frac{\partial \phi}{\partial x_k}\right).$$

All the theorems previously stated for distribution in one dimensions also hold for distributions in n dimensions.

Besides the distributions we have introduced above, we shall also consider the direct product of a distribution in one space and a distribution in another space. In order to explain this concept, we shall work with a simple illustration.

Suppose that S_x is a distribution on the one-dimensional space of testing functions $\phi(x)$ while T_y is a distribution on the one-dimensional space of testing functions $\psi(y)$. From the one-dimensional testing functions $\phi(x)$ and $\psi(y)$ we may form two-dimensional testing functions $\theta(x,y)$ by the following device:

$$\theta(x,y) = \sum_{i=1}^{n} \phi_i(x)\psi_i(y). \tag{2.54}$$

It is easy to show* that every two-dimensional testing function of x and y can be approximated by sums such as that in (2.54).

We now show that we may define a two-dimensional distribution W_{xy} on the space of two-dimensional testing functions $\theta(x,y)$. This distribution has the property that

$$W_{xy}\left[\phi(x)\psi(y)\right] = S_x(\phi)T_y(\psi). \tag{2.55}$$

Consider any two-dimensional testing function $\theta(x,y)$ as a function of x depending on a parameter y. For fixed y, $\theta(x,y)$ is a testing function of x and then

$$S_x\left[\theta(x,y)\right] = \Psi(y)$$

is a function of y. From Theorem X it follows that $\Psi(y)$ is a testing function in y and we may then define

$$T_y\left[\Psi(y)\right] = T_y\left[S_x\left[\theta(x,y)\right]\right] = W_{x,y}(\theta).$$

It is clear that

$$W\left[\phi(x)\psi(y)\right] = T_y\left[S_x\left[\phi(x)\psi(y)\right]\right] = T_y\left[\psi S_x(\phi)\right] = T_y(\psi)S_x(\phi).$$

The above construction shows that there exists a distribution W having the property (2.55). We call W the direct product of S and T. This distribution must be unique since it has a unique value on every sum of the form (2.54).

Note also that W could just as well be defined by the formula:

$$W\left[\theta(x,y)\right] = S_x\left[T_y(\phi(x,y)\right].$$

Since W is unique, this imples that

---

* L. Schwartz, Theorie des Distributions, op.cit., tome 1, p.107.

$$S_x\left[T_y[\theta(x,y)]\right] = T_y\left[S_x[\theta(x,y)]\right]$$

or, expressed in terms of symbolic functions,

$$(2.56) \qquad \int s(x)dx \int t(y)\theta(x,y)dy = \int t(y)dy \int s(x)\theta(x,y)dx.$$

We may summarize our results in

Theorem XI. <u>If S is a distribution over the space of testing functions $\phi(x)$ and T is a distribution over the space of testing functions $\psi(y)$, then there exists a unique distribution W over the space of two-dimensional testing functions $\theta(x,y)$ with the property that</u>

$$W|\phi(x)\psi(y)| = S(\phi)\ T(\psi)\ .$$

<u>Also,</u>

$$W[\theta(x,y)] = S_x\left[T_y[\theta(x,y)]\right] = T_y\left[S_x[\theta(x,y)]\right].$$

Note that the last result or what is equivalent, formula (2.47), amounts to a statement of the fact that we may interchange the order of integration of symbolic functions.

The concept of direct product of distributions can obviously be extended to the case where we start with distributions defined on m- and n-dimensional space, and to the case where we have more than two factors. We leave the details of these cases to the reader.

<u>δ-Functions in Different Coordinate Systems</u>

The n-dimensional δ-function $\delta(x_1,x_2,\ldots,x_n)$ is the symbolic function for the distribution which assigns the value $\phi(0,0,\ldots,0)$ to the testing function $\phi(x_1,x_2,\ldots,x_n)$. Since the n-dimensional δ-function applied to the product of one dimensional testing functions $\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)$ gives the value $\phi_1(0)\phi_2(0)\ldots\phi_n(0)$, we see from Theorem XI that

$$(2.57) \qquad \delta(x_1,x_2,\ldots,x_n) = \delta(x_1)\delta(x_2)\ldots\delta(x_n).$$

Consider now the special case of two dimensional space and suppose that we change coordinates from x_1,x_2 to ξ_1, ξ_2 using the formulas:

$$x_1 = u(\xi_1, \xi_2)$$

$$x_2 = v(\xi_1, \xi_2).$$

Here we assume that u and v are continuous differentiable one-valued functions in the ordinary sense and we also assume that $\xi_1 = \beta_1$, $\xi_2 = \beta_2$ corresponds to $x_1=\alpha_1, x_2=\alpha_2$. The integral

$$\int\int \delta(x_1-\alpha_1)\delta(x_2-\alpha_2)\phi(x_1,x_2)dx_1dx_2 = \phi(\alpha_1,\alpha_2),$$

by the change of coordinates, becomes

$$(2.58)\qquad \iint \delta\left[u(\xi_1,\xi_2)-\alpha_1\right]\delta\left[v(\xi_1,\xi_2)-\alpha_2\right]\,\phi(u,v)|J|\,d\xi_1 d\xi_2 = \phi(\alpha_1,\alpha_2),$$

where J, the Jacobian of the coordinate transformation, is given by the formula

$$J = \frac{\partial(u,v)}{\partial(\xi_1,\xi_2)}.$$

Equation (2.58) shows that the symbolic function

$$\delta\left[u(\xi_1,\xi_2),\ v(\xi_1,\xi_2)\right]\ |J|$$

assigns to any testing function ξ_1, ξ_2, the value of that function at the points where $u = \alpha_1$, $v = \alpha_2$, or, what is the same, where $\xi_1 = \beta_1$, $\xi_2 = \beta_2$; consequently, we may write

$$\delta\left[u(\xi_1,\xi_2)-\alpha_1\ \ v(\xi_1,\xi_2)-\alpha_2\right]\ |J| = \delta(\xi_1-\beta_1)\delta(\xi_2-\beta_2)$$

or, finally, if $J \neq 0$

$$(2.59)\qquad \delta(x_1-\alpha_1)\delta(x_2-\alpha_2) = \frac{\delta(\xi_1-\beta_1)\delta(\xi_2-\beta_2)}{|J|}.$$

As an illustration of this theorem consider the transformation from rectangular coordinates x,y to polar coordinates r,θ. If $\phi(x,y)$ is a testing function, then

$$\iint \delta(x-x_0)\delta(y-y_0)\phi(x,y)dxdy = \phi(x_0,y_0).$$

Suppose that the point with rectangular coordinates x_0, y_0 has polar coordinates r_0, θ_0 and suppose that $\bar{\phi}(r,\theta)$ is $\phi(x,y)$ expressed in polar coordinates. Since

$$\iint \frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r}\ \bar{\phi}(r,\theta)\,r\,dr\,d\theta = \bar{\phi}(r_0,\theta_0),$$

and since $\bar{\phi}(r_0,\theta_0) = \phi(x_0,y_0)$, it is clear that

$$(2.60)\qquad \delta(x-x_0)\delta(y-y_0) = \frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r}.$$

This is exactly the result that would be obtained by an application of formula (2.59).

We now investigate what happens to formula (2.59) when J = 0. In such a case the transformation from x_1, x_2 to ξ_1, ξ_2 breaks down and is no longer one-to-one. For example, in the transformation from rectangular coordinates to polar coordinates, the Jacobian becomes zero at the origin x = y = 0 and we know that there the transformation is no longer one-valued since r = 0 and θ may have any value. This situation is typical of the general case. We shall call a coordinate such as θ, which is either many-valued or else has no determinate value at a singular point of the transformation, an ignorable coordinate.

Suppose that $x_1=\alpha_1$, $x_2=\alpha_2$ is a singular point of the transformation, that ξ_2 is an ignorable coordinate there, and suppose that the testing function $\phi(x_1,x_2)$ in x_1,x_2 space becomes the testing function $\bar{\phi}(\xi_1,\xi_2)$ in $\xi_1\,\xi_2$ space; then $x_1=\alpha_1$, $x_2=\alpha_2$ corresponds to $\xi_1 = \beta_1$, ξ_2 indeterminate, and also $\phi(\alpha_1,\alpha_2)$ is a function $\bar{\phi}(\beta_1)$ only of β_1.

The symbolic function corresponding to $\delta(x_1-\alpha_1)\delta(x_2-\alpha_2)$ will consequently be a symbolic function $t(\xi_1)$ only of ξ_1. Formula (2.58) then becomes

$$(2.61)\qquad \phi(\alpha_1,\alpha_2) = \int\int \delta(u-\alpha_1)\delta(v-\alpha_2)\phi(u,v)\ |J|\ d\xi_1 d\xi_2$$

$$= \int\int t(\xi_1)\bar{\phi}(\xi_1,\xi_2)|J|d\xi_1,d\xi_2 = \bar{\phi}(\beta_1).$$

Put

$$\int |J|d\xi_2 = J_1;$$

then (2.60) will be satisfied if we put

$$t(\xi_1) = \frac{\delta(\xi_1-\beta_1)}{J_1} \quad ;$$

consequently, in case $J = 0$, when $x_1 = \alpha_1$, $x_2 = \alpha_2$, we have

$$(2.62)\qquad \delta(x_1-\alpha_1)\delta(x_2-\alpha_2) = \frac{\delta(\xi_1-\beta_1)}{J_1} \quad .$$

For example, in the transformation from rectangular to polar coordinates, θ is an ignorable coordinate at the origin, and

$$\int_0^{2\pi} rd\theta = 2\pi r;$$

consequently,

$$\delta(x)\delta(y) = \frac{\delta(r)}{2\pi r} \quad .$$

Similar results hold for transformations in n-dimensional space. We state then in

Theorem XII. *Let $x_1,x_2,\ldots,x_n$ be n-dimensional rectangular coordinates and let $\xi_1, \xi_2,\ldots, \xi_n$ be any other coordinate system, not necessarily rectangular, with n-dimensional volume element $Jd\xi_1 d\xi_2\ldots d\xi_n$. Suppose that the point P with coordinates $x_1=\alpha_1, x_2=\alpha_2,\ldots,x_n=\alpha_n$ has coordinates $\xi_1=\beta_1,\ldots,\xi_n=\beta_n$ and that $J \neq 0$ at P; then*

$$(2.63)\qquad \delta(x_1-\alpha_1)\ \ldots\ \delta(x_n-\alpha_n) = \frac{\delta(\xi_1-\beta_1)\ldots\delta(\xi_n-\beta_n)}{J} \quad .$$

If $J = 0$ at the point P, suppose that the equations $\xi_1 = \beta_1,\ldots,\xi_k=\beta_k$ define P and therefore the coordinates $\xi_{k+1},\ldots\xi_n$ are ignorable. Put

$$J_k = \int\ldots\int Jd\xi_{k+1}\ \ldots\ d\xi_n,$$

so that J_k is the integral of the Jacobian over the ignorable coordinates; then

$$(2.64)\qquad \delta(x_1-\alpha_1)\ldots\delta(x_n-\alpha_n) = \frac{\delta(\xi_1-\beta_1)\ldots\delta(\xi_k-\beta_k)}{J_k} \quad .$$

The proof of this theorem will be left to the reader. We shall illustrate it by considering the transformation from three-dimensional rectangular coordinates x,y,z to spherical coordinates r,θ,ϕ. Here $J = r^2\sin\theta$ which vanishes for points on

on the polar axis where ϕ is ignorable and also at the origin where both θ and ϕ are ignorable. If p has coordinates x_o, y_o, z_o in rectangular coordinates and r_o, θ_o, ϕ_o in spherical coordinates, then

$$\delta(x-x_o)\delta(y-y_o)\delta(z-z_o) = \frac{\delta(r-r_o)\delta(\theta-\theta_o)\delta(\phi-\phi_o)}{r^2\sin\theta}, \text{ when } r_o\neq 0,\ \theta_o\neq 0$$

$$= \frac{\delta(r-r_o)\delta(\theta)}{2\pi r^2\sin\theta}, \text{ when } x_o=y_o=0,\ z_o\neq 0 \text{ or } r_o\neq 0, \theta_o=0, \tag{2.65}$$

$$= \frac{\delta(r)}{4\pi r^2}, \text{ when } x_o=y_o=z_o=0 \text{ or } \theta_o = r_o = 0.$$

Problems

2.26 Verify the relations in equation (2.65) by operating with both sides in testing functions.

2.27 Show that in two dimensions

$$\frac{\delta(r-r_o)}{r} = \int_0^{2\pi} \delta(x-r_o\cos\theta)\delta(y-r_o\sin\theta)d\theta .$$

Hint. Integrate equation (2.60) with respect to θ_o.

2.28 Verify formula (2.65) by ap lying it to a testing function which is equal to e^{-x-y-z} in the sphere $x^2+y^2+z^2 = 1$.

2.29 Verify Theorem XII by applying each side of (2.63) and (2.64) to a testing function.

Fourier Integrals

The Fourier integral for a distribution T is defined to be $T_k(e^{-ikx})$ if it exists. The theory is complicated because of the fact that e^{-ikx} is not a testing function. However, we can approximate e^{-ikx} by a sequence of testing functions $\phi_n(k,x)$ which vanish outside an interval whose length goes to infinity with n, and then we define

$$T_k(e^{-ikx}) = \lim T_k(\phi_n), \tag{2.66}$$

where the limit is in the sense of distributions. Of course if T has a compact support, the limit in (2.66) and therefore also the Fourier integral always exist. For example, if T is the δ-function its Fourier integral is

$$\int \delta(k)e^{-ikx}dk = 1.$$

If 1 is considered as defining a distribution, its Fourier integral will be defined by (2.66). The right side of (2.54) is essentially

$$\lim_{n\to\infty} \int_{-n}^{n} e^{ikx}dk.$$

This limit may be obtained by using the theory of the Dirichlet kernel* but we see that

* See Titchmarsh, Theory of Functions op.cit..

it is merely the case n = 0 of formula (2.39). We have then

$$(2.67) \qquad \int e^{-ikx}dk = 2\pi\delta(x)$$

or, replacing x by x-x', we get

$$\int e^{-ik(x-x')}dk = 2\pi\delta(x-x').$$

Consider an arbitrary distribution T and take the fold of (2.67) with T. Using symbolic functions, we have

$$\int t(x)dx \int e^{-ik(x-x')}dk = 2\pi\int t(x)\delta(x-x')dx = 2\pi t(x');$$

but with the help of Theorem XI we may interchange the order of integration and find that

$$(2.68) \qquad 2\pi t(x') = \int e^{ikx'}dk \int t(x)e^{-ikx}dx.$$

The inner integral is exactly the Fourier integral of the distribution T_x as we have defined it in (2.66), but now we consider T as acting on functions of x and not on functions of k.

Put

$$(2.69) \qquad T(k) = \int t(x)e^{-ikx}dx$$

where, as always, the right side is to be understood in the sense indicated in (2.66). We shall call T(k) the Fourier transform of t(x). Formula (2.68) now gives an inversion theorem for the Fourier transform, namely

$$(2.70) \qquad t(x) = \frac{1}{2\pi}\int T(k)e^{ikx}dk.$$

Note this inversion formula is the Fourier transform of T(k) except for the change of sign of the imaginary unit and the factor $1/2\pi$.

We shall illustrate the inversion formula by considering the Fourier integral given in (2.39). It states that

$$\int x^n e^{-ikx}dx = 2\pi i^n\delta^{(n)}(k),$$

consequently the Fourier transform of x^n is $2\pi i^n\delta^{(n)}(k)$.

Now, the inversion formula would give the following:

$$x^n = \frac{1}{2\pi}\int 2\pi i^n\delta^{(n)}(k)\ e^{ikx}dk.$$

This formula is clearly correct since

$$\int \delta^{(n)}(k)\ \phi(k)dk = (-)^n\phi^{(n)}(0).$$

It should also be noticed that as a consequence of (2.39) every polynomial has a Fourier transform and that this transform is a linear combination of derivatives of the delta functions. We state without proof that the fact that every integrable function f(x) which satisfies the inequality

$$|f(x)| < C\, e^{\epsilon|x|}$$

for all $\epsilon > 0$ and all x sufficiently large in absolute value, has a Fourier transform. This Fourier transform will in general be a distribution. We shall not discuss under what conditions a distribution has a Fourier transform. In applications it is usually not difficult to decide whether the Fourier transform exists. If it does exist, the inversion formulas (2.69) and (2.70) will hold.

Problem:

2.30 Prove that

$$\int \operatorname{sgn} x\, e^{-ikx} dx = -\frac{2i}{k},$$

and prove also the inverse formula

$$\frac{1}{2\pi}\int \frac{-2ie^{ikx}dk}{k} = \operatorname{sgn} x.$$

Properties of the Fourier Transform

We assume that all distributions considered in this section have Fourier transforms. The first property we shall discuss is given in

Theorem XIII. If T is a distribution with compact support, then the Fourier transform of T is an entire analytic function.

Suppose that the support of T is contained in the interval $-R \le x \le R$. Let $\phi_k(x)$ be any testing function which is identically equal to e^{-ikx} for $-R \le x \le R$; then the Fourier transform of T is given by the following formula which is valid for all real and complex k.

$$\mathcal{T}(k) = T(e^{-ikx}) = T(\phi_k).$$

By Theorem X the derivative of $\mathcal{T}(k)$ is

$$\mathcal{T}'(k) = \frac{d}{dk} T(\phi_k) = T\left(\frac{d}{dk}\phi_k\right) = -iT(xe^{-ikx})$$

which again exists for all real and complex k. Since the derivative of $\mathcal{T}(k)$ exists and is unique for all complex values of k, it is proven that $\mathcal{T}(k)$ is an entire analytic function of k.

Note that since

$$T(x\phi) = xT(\phi),$$

we have also proved that the Fourier transform of xT is $i\,\mathcal{T}'(k)$. From this it follows easily that the Fourier transform of $x^n T$ is $i^n \mathcal{T}^{(n)}(k)$. Conversely, it can be shown that the Fourier transform of $T^{(n)}$ is $i^n k^n \mathcal{T}(k)$. We re-state these properties of the Fourier transform as

Theorem XIV. If T is a distribution which has a Fourier transform $\mathcal{T}(k)$, then $x^n T$ has Fourier transform $i^n \mathcal{T}^{(n)}(k)$ and $T^{(n)}$ has Fourier transform $i^n k^n \mathcal{T}(k)$.

The proof above used the fact that T has a compact support, but it is easy to show that this restriction is unnecessary.

The last property of the Fourier integral which we shall discuss is the convolution or folding property. We have

Theorem XV. Suppose that T and S are any two distributions one of which has a compact support, and suppose that the Fourier transform of T is $\tau(k)$ and the Fourier transform of S is $\sigma(k)$; then the Fourier transform of the fold of T and S is the product $\tau(k)\,\sigma(k)$.

Note that the product $\tau(k)\,\sigma(k)$ always makes sense since by Theorem XIII and the hypothesis one of the functions is analytic.

The proof of Theorem XV is easy. By definition the fold of T and S is

$$T*S(\phi) = T_x[S_y[\phi(x+y)]]$$

and by definition the Fourier transform of the fold is

$$T*S(e^{-ikx}) = T_x[S_y(e^{-ikx-iky}]] = T_x(e^{-ikx})S_y(e^{-iky}) = \tau(k)\,\sigma(k).$$

This proves the theorem.

Problems:

2.31 Find the Fourier transform of $|x|$. Use the result of Problem 2.30 to show that the result is i times the derivative of the Fourier transform of sgn x. Compare this with Theorem XIV.

2.32 Prove Theorem XIV for an arbitrary distribution that has a Fourier transform.

2.33 Prove the converse of Theorem XV: If the product of the distributions T and S has a meaning and if T and S have Fourier transforms $\tau(k)$ and $\sigma(k)$ respectively, then the Fourier transform of the product TS is the fold of $\tau(k)$ and $\sigma(k)$.

2.34 Use the result of Problem 2.33 and formula (2.50) to prove Theorem XIV again.

Fourier Transforms in Three Dimensions

The Fourier integral and the inversion theorem can be extended to distributions in n-dimensions. We use the n-dimensional equivalent of (2.67), namely

$$\int\cdots\int \exp[-i(k_1x_1+\ldots k_nx_n]]\,dk_1\ldots dk_n = (2\pi)^n\delta(x_1)\ldots\delta(x_n).$$

Replace x_i by $x_i-\xi_i$ for i = 1,2,.. ,n and we get

$$\int\cdots\int \exp[-i\{k_1(x_1-\xi_1)+\ldots+k_n(x_n-\xi_n)\}]\,dk_1\ldots dk_n = (2\pi)^n\delta(x_1-\xi_1)\ldots\delta(x_n-\xi_n).$$

This formula can be used in the same way as the corresponding formula in one dimension to prove that if $t(x_1,x_2,\ldots,x_n)$ is an n-dimensional function (symbolic or not) and if we define the Fourier transform by

$$\tau(k_1,\ldots,k_n)=\int\cdots\int t(x_1,\ldots,x_n)\exp[-i(k_1x_1+\ldots+k_nx_n]]\,dx_1\ldots dx_n\ , \tag{2.71}$$

then the following inversion theorem holds:

$$(2.72)\qquad t(x_1,\ldots,x_n)=(2\pi)^{-n}\int\cdots\int\tau(k_1,\ldots,k_n)\exp\left[i(k_1x_1+\ldots+k_nx_n)\right]dk_1\ldots dk_n.$$

The three dimensional case is of particular interest. Put n=3, let x_1,x_2,x_3 be the components of a vector $\underline{r}$ and let k_1,k_2,k_3 be the components of a vector $\underline{k}$; then (2.71) and (2.72) become

$$(2.73)\qquad \tau(\underline{k}) = \int t(\underline{r})e^{-i\underline{k}\underline{r}}\,d\underline{r}$$

$$t(\underline{r}) = (2\pi)^{-3}\int (\underline{k})e^{i\underline{k}\cdot\underline{r}}\,d\underline{k}$$

Here $d\underline{r}$ and $d\underline{k}$ are the volume elements in three-dimensional space and $\underline{k}.\underline{r}$ indicates the scalar product of the vectors $\underline{k}$ and $\underline{r}$.

The properties of the Fourier transform discussed in the previous section can be suitably generalized so that they are valid for the transforms in (2.73). We shall show how this can be done for the convolution or folding property. Let $t(\underline{r})$ and $s(\underline{r})$ be functions which have Fourier transforms $\tau(\underline{k})$ and $\sigma(\underline{k})$ respectively. The fold of $t(\underline{r})$ and $s(\underline{r})$ is

$$\int t(\underline{r}')s(\underline{r}-\underline{r}')d\underline{r}'.$$

The Fourier transform of the fold is

$$\int e^{-i\underline{k}\cdot\underline{r}}d\underline{r}\int t(\underline{r}')s(\underline{r}-\underline{r}')d\underline{r}' = \int e^{-i\underline{k}\cdot\underline{r}'}t(\underline{r}')d\underline{r}'\int e^{-ik\cdot(\underline{r}-\underline{r}')}s(\underline{r}-\underline{r}')d\underline{r}$$

$$= \int e^{-i\underline{k}\cdot\underline{r}'}t(\underline{r}')d\underline{r}'\int e^{-i\underline{k}\cdot\underline{\xi}}s(\underline{\xi})d\xi = \tau(\underline{k})\,\sigma(\underline{k});$$

consequently, we have the same result as in the one-dimensional case, namely, the Fourier transform of a fold is the product of the Fourier transforms.

In many applications $t(\underline{r})$ is a function solely of the length r of the vector $\underline{r}$. In that case the formula (2.73) for the Fourier transform will be simplified by the introduction of spherical polar coordinates r,θ,ϕ. The angles θ, ϕ are defined relative to a polar axis which, for simplicity, we take to be along the direction of the vector $\underline{k}$. Formula (2.73) now becomes

$$\tau(\underline{k}) = \int_0^\infty\int_0^{\pi/2}\int_0^{2\pi} t(r)e^{-ikr\cos\theta}r^2\sin\theta\,dr\,d\theta\,d\phi$$

where k is the length of the vector $\underline{k}$. If the integration with respect to θ and ϕ is carried out, we obtain

$$(2.74)\qquad \tau(\underline{k}) = \tau(k) = 4\pi\int_0^\infty t(r)\,\frac{\sin kr}{kr}\,r^2dr.$$

Similarly, we obtain the inverse formula:

$$(2.75)\qquad t(r) = \frac{1}{2\pi^2}\int_0^\infty \tau(k)\,\frac{\sin kr}{kr}\,k^2dk.$$

Problems

2.35 Find the three-dimensional Fourier transform of r^n where $r = (x^2+y^2+z^2)^{1/2}$. Hint. Separate Problem (2.20) into real and imaginary parts to obtain the required integral.

2.36 Show that the integral over six dimensions

$$\int\int f(r_1)g(|\underline{r}_1-\underline{r}_2|)f(r_2)d\underline{r}_1 d\underline{r}_2$$

$$= \int |\alpha(k)|^2 \beta(k) d\underline{k}$$

where $\alpha(k)$, $\beta(k)$ are the Fourier transforms of $f(r)$ and $g(r)$ respectively. Hint. Replace $g(\underline{r}_1-\underline{r}_2)$ be an integral in terms of its Fourier transform $\beta(k)$ as follows:

$$g(\underline{r}_1-\underline{r}_2) = \int e^{i\underline{k}\cdot(\underline{r}_1-\underline{r}_2)}\beta(k)d\underline{k}.$$

2.37 Use Problem 2.36 to evaluate the following six-dimensional integral:

$$\int\int e^{-r_1} e^{-r_2} |\underline{r}_1-\underline{r}_2| dr_1 dr_2 .$$

Evaluating Finite Part Integrals

The concept of the finite part of integrals which we introduced previously (see page 80) plays a very useful role in the applications of Fourier integrals. Before discussing these applications we shall show how in certain cases the finite part of integrals may be evaluated by means of complex integration.

Consider the finite part of the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ikx}dk}{('k)^n}$$

where n is a positive integer. I is the limit as R approaches infinity of the integral

$$I_R = \int_{-R}^{R} \frac{e^{-ikx}}{('k)^n} dk.$$

Let us investigate the value of the integral of $\frac{e^{-ikx}}{k^n}$ over the contour C in the complex k-plane illustrated in Figure 1. The contour C is made up of two semi-

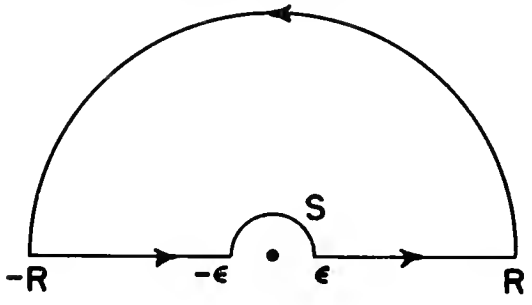

Figure 1

circles, one of radius, R, the other of radius ϵ, and two segments of the real k-axis, namely from -R to $-\epsilon$ and from ϵ to R.

Since the integrand is regular inside $\mathcal{C}$, then by Cauchy's theorem the integral over $\mathcal{C}$ will vanish. If x is negative, the integral over the semi-circle of radius R will vanish as R approaches infinity; we have then

$$0 = \lim_{R\to\infty}\left[\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right]\frac{e^{-ikx}}{k^n}\,dk + \int_S \frac{e^{-ikx}}{k^n}\,dk. \tag{2.76}$$

The last term is the integral over the small semi-circle from $-\epsilon$ to ϵ.

Notice that the integrals in (2.76) are ordinary integrals and not only the finite parts. Of course, as ϵ approaches zero each term in (2.76) becomes infinite, but their sum remains zero. We conclude then that the finite part of the first term is equal to the negative of the finite part of the second; consequently,

$$I = -\int_S \frac{e^{-ikx}dk}{(ik)^n}.$$

This last integral may be evaluated by expanding the numerator in powers of x. We have

$$I = -\sum_{j=0}^{\infty}(-ix)^j\int_S \frac{k^j dk}{(ik)^n}.$$

Now, as ϵ approaches zero consider the limit of the integral

$$\int_S \frac{k^j}{(ik)^n}\,dk$$

When j is less than n-1, the limit is infinity; when j = n-1, the limit is $-\pi i$; when j is greater than n-1, the limit is zero; consequently,

$$I = \pi i\,(-ix)^{n-1}$$

for x negative.

If x is positive, we use a similar contour with semi-circles in the lower half-plane and we find that

$$I = -\pi i(-ix)^{n-1}.$$

We may write both results in the form

$$I = \pi x^{n-1}(-i)^n \operatorname{sgn} x. \tag{2.76a}$$

As another illustration consider

$$\int_0^{\infty}\frac{e^{-ikx}dk}{i(k-k_0)}, \qquad k_0 > 0.$$

Replace the integral along the real axis by the integral over the contour $\mathcal{C}$ illustrated in Figure 2.

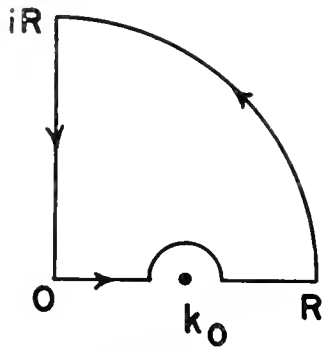

Figure 2

Since there are no singularities inside C, the integral over C is zero. For x negative, the integral over the large semi-circle vanishes. The integral over the small semi-circle is $-\pi i\, e^{-ik_o x}$; consequently, for $x < 0$ we have

$$\int_o^\infty \frac{e^{-ikx}dk}{(k-k_o)} = \pi i\, e^{-ik_o x} + \int_o^\infty \frac{e^{tx}dt}{t+ik_o}$$

The last term is the contribution from the integral over the imaginary axis where we have put k = it. For x positive we use a similar contour in the lower half-plane and we obtain

$$\int_o^\infty \frac{e^{-ikx}dk}{(k-k_o)} = -\pi i\, e^{-ik_o x} - \int_o^\infty \frac{e^{-tx}dt}{t-ik_o}.$$

Similar methods can be applied to

$$J = \int_o^\infty \frac{\cos kx\, dk}{(k^2-k_o^2)}.$$

We first write the integrals in terms of exponentials as follows:

$$J = \frac{1}{2}\int_o^\infty \frac{e^{ikx}+e^{-ikx}}{(k^2-k_o^2)}dk = \frac{1}{2}\int_o^\infty \frac{e^{ikx}}{(k^2-k_o^2)}dk$$

This last integral is replaced by a contour integral indented at the singularities and closed in the upper or lower half-plane according as x is positive or negative. Evaluating the contributions at the singularities we find that

$$J = -\frac{\pi}{2k_o}\sin(k_o|x|).$$

As a final illustration consider the integral

$$K_\nu = \int_{-\infty}^\infty \frac{e^{-ikx}dk}{(k)^\nu},$$

where ν is an arbitrary positive member. In the beginning of this section we have evaluated the integral when ν is an integer. If ν is not an integer then the integral K will not be completely defined until we specify which branch of the multi-valued function, k^ν we are using in the integral. For example, if $\nu = 1/2$ then when k is positive, $k^{1/2}$ may be taken as the positive square root; but when k is negative, does

$k^{1/2}$ mean the positive imaginary or the negative imaginary square root? Suppose we specify that $k^{1/2}$ should mean the positive imaginary square root. Then we may evaluate $K_{1/2}$ by the previous methods if we take into account the fact that, because of our specification of the value of $k^{1/2}$, the lower half of the complex k-plane must contain a branch cut. When x is negative, the contour may be closed in the upper half-plane thus avoiding the branch cut, and we find $K_{1/2} = 0$. When x is positive, the appropriate contour is illustrated in Figure 3. The shaded line along the negative imaginary axis represents the branch cut. The integrals over the semi-circles vanish but the integrals over the branch cuts do not vanish.

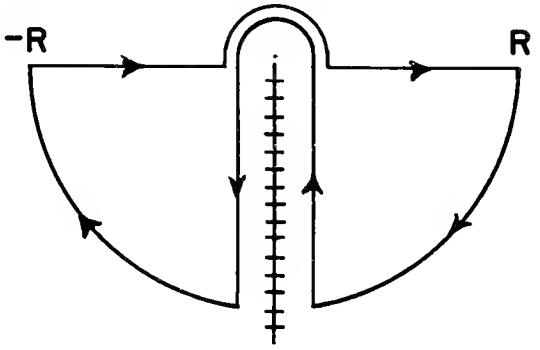

Figure 3

The integral on the left side of the cut is equal, except for a factor -1, to the integral on the right side of the cut. Replace k by -it in the branch cut integrals and we have

$$K_{1/2} = 2e^{-\pi i/4} \int_0^\infty \frac{e^{-tx}dt}{t^{1/2}} = 2\sqrt{\pi}\, e^{-\pi i/4}.$$

for $x > 0$.

A similar treatment must be used for other values of ν. It may seem unnecessary when $\nu = 1/3$ say, for then it would seem that $k^{1/3}$ is positive when k is positive and negative when k is negative. However, with this specification the function $k^{1/3}$ is no longer continuous in the complex k-plane. This may be seen by following the values of $k^{1/3}$ along the unit circle k = +1 to k= -1. If we start at k = 1 with $k^{1/3} = 1$, then if we go along the unit circle in the upper half-plane we get $(-1)^{1/3} = e^{\pi i/3}$ but if we go along the unit circle in the lower half-plane we get $(-1)^{1/3} = e^{-\pi i/3}$. In neither case can we get $(-1)^{1/3} = -1$.

Nevertheless, despite the fact that $k^{1/3}$ is not continuous in the complex k-plane, the value of $K_{1/3}$ may be obtained, but we must proceed slightly differently. Since $k^{1/3}$ is an odd function of k, we have

$$\int_{-\infty}^{\infty} \frac{e^{-ikx}dk}{k^{1/3}} = -2i \int_0^\infty \frac{\sin kx}{k^{1/3}}\, dk = \int_0^\infty \frac{e^{-ikx} - e^{ikx}}{k^{1/3}}\, dk.$$

Let us put the branch cut of $k^{1/3}$ along the negative real axis; then one term of the integral may be rotated in the upper half-plane until it lies along the upper imaginary axis while the other is rotated in the lower half-plane until it also lies along the lower imaginary axis. **In this way we find that if $x > 0$,**

$$K_{1/3} = \int_0^{-i\infty} \frac{e^{-ikx}dk}{k^{1/3}} - \int_0^{i\infty} \frac{e^{ikx}}{k^{1/3}}\,dk.$$

Put $k = -it$ and $k = it$ respectively in these integrals and we have

$$K_{1/3} = (e^{-\frac{\pi i}{3}} - e^{\frac{\pi i}{3}}) \int_0^{\infty} \frac{e^{-kx}dk}{k^{1/3}} \quad .$$

The last integral can be found from tables of the Gamma Function.

From these examples we may derive the following method for finding the finite part of an integral:

Method. Replace the integral by a contour integral indented at the singularities and closed in the upper or lower half-plane, and evaluate the contour integral by the theory of residues.

It may have been noticed that the integrals we have evaluated in this **section have been essentially Fourier transforms. For example, (2.76a) shows that the** Fourier transform of $(\mathrm{i}k)^{-n}$ is $\pi(-\mathrm{i})^n x^{n-1}\operatorname{sgn} x$. We shall verify this result for the cases $n = 0$ or $n = 1$, by using the inverse Fourier transform to show that the Fourier transform of $\operatorname{sgn} x$ is $\frac{\mathrm{i}}{\pi \mathrm{i}k}$ and that the Fourier transform of $|x|$ is $-\frac{1}{\pi(\mathrm{i}k)^2}$.

Consider the integrals

$$I_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{sgn} x\, e^{\mathrm{i}kx}dx$$

and

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |x| e^{\mathrm{i}kx}dx.$$

We have

$$I_0 = \frac{2\mathrm{i}}{2\pi}\int_0^{\infty} \sin kx\, dx = \frac{2\mathrm{i}}{2\pi \mathrm{i}k} = \frac{\mathrm{i}}{\pi \mathrm{i}k}$$

and

$$I_1 = \frac{2}{2\pi}\int_0^{\infty} x \cos kx\, dx = \frac{1}{\pi}\frac{d}{dk}\int_0^{\infty} \sin kx\, dx = -\frac{1}{\pi(\mathrm{i}k)^2},$$

which agrees with the results above.

Before concluding this section, we wish to point out that caution is needed when transforming variables in a finite part integral. Consider

$$\int_0^2 \frac{dx}{\mathrm{i}x} = \log 2.$$

If we make the transformation $y = ax$, the integral becomes

$$\int_0^{2a} \frac{dy}{'y} = \log 2a$$

instead of log 2. In order to avoid this difficulty we make the following restriction:

Restriction: When the variable of integration in a finite part integral is changed, the value of the derivative of the new variable with respect to the original variable must be 1 at all the singular points of the integrand.

As an illustration, consider the integral

$$\int_0^{\pi/3} \cot\theta \, d\theta = \log \sin\pi/3 = \frac{1}{2}\log(3/4).$$

Suppose we wish to make the transformation $u = \tan\theta/2$ in order to obtain the integral of a rational function. This transformation will change the value of the integral. To satisfy the restriction, we must change the transformation to $u = 2\tan\theta/2$, and then the integral becomes

$$\int_0^{2(3)^{-1/2}} \frac{1-\frac{u^2}{4}}{u} \frac{du}{1+\frac{u^2}{4}} = \left[\log u - \frac{1}{2}\log(1+\frac{u^2}{4})\right]$$

$$= \frac{1}{2}\log(3/4),$$

which agrees with the original evaluation.

Principal Value Integrals and Singular Integral Equations.

We have seen (page 88) that

$$\lim_{a\to 0} \frac{1}{x+ia} = \frac{1}{'x} - i\pi\delta(x).$$

Similarly, we can prove that

$$\lim_{\epsilon\to 0} \frac{1}{x-i\epsilon} = \frac{1}{'x} + i\pi\delta(x).$$

This result has interesting consequences in the theory of analytic functions. Suppose that $w(z)$ is an analytic function of z in the upper half-plane and that $w(z)$ vanishes sufficiently fast at infinity so that an application of Cauchy's theorem gives

$$w(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\phi(x)dx}{x-z}.$$

Put $z = a+i\epsilon$ and consider the limit of the integral as ϵ approaches zero. We have

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(x)dx}{x-a-i\epsilon} = \frac{1}{2\pi i}\int \frac{w(x)}{'(x-a)}dx + \frac{1}{2}w(a) = w(a);$$

consequently, the value of $w(z)$ on the boundary, that is, the real axis, is given by the principal value integral

$$(2.77)\qquad w(a) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{w(x)dx}{'(x-a)} \quad .$$

Now, suppose that u(z) and v(z) are the real and imaginary parts respectively, of the analytic function w(z). Putting w = u+iv into the above formula and separating into real and imaginary parts, we have

$$(2.78)\qquad u(a) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(x)dx}{'(x-a)}\ ,\quad v(a) = -\frac{1}{\pi}\int \frac{u(x)dx}{'(x-a)}\ .$$

These formulas show that the real part of an analytic function can be found from a knowledge of the imaginary part, and conversely, the imaginary part can be found from the real part. The methods of the last section can be used to verify this result in the case of simple functions such as $w(z) = e^z$.

The formulas (2.78) can be written very simply in terms of the folding concept that was introduced on page 94. We have

$$(2.79)\qquad u(x) = \frac{1}{\pi} v * ('x)^{-1},\quad v = -\frac{1}{\pi} u * ('x)^{-1}\ .$$

If we substitute the first formula of (2.79) into the second we get

$$v = -\frac{1}{\pi^2}\left(v * ('x)^{-1}\right) * ('x)^{-1} = -\frac{1}{\pi^2} v * ('x)^{-1} * ('x)^{-1}.$$

Since this formula is valid for arbitrary functions v(x) we must have

$$(2.80)\qquad ('x)^{-1} * ('x)^{-1} = -\pi^2\,\delta(x),$$

or, expressed in words, the fold of the principal value $1/x$ with itself is $-\pi^2\delta(x)$.

Another proof of (2.80) may be obtained by using the Fourier integral. By Theorem XV the Fourier transform of a fold is the product of the Fourier transforms of the components; consequently, since the Fourier transform of $('x)^{-1}$ is πi sgn k (see the previous section), we find that the Fourier transform of the fold of $('x)^{-1}$ with itself is $-\pi^2(\mathrm{sgn}\,k)^2 = -\pi^2$. This result is obviously the Fourier transform of $-\pi^2\delta(x)$; therefore we again obtain (2.80).

We shall now show how (2.80) may be used to solve a singular integral equation which is important in aerodynamic theory. Consider the following integral equation for the unknown function u(x):

$$u(x) - \lambda\int \frac{u(y)dy}{'(x-y)} = f(x)$$

or

$$(2.81)\qquad u(x) - \lambda\, u*('x)^{-1} = f.$$

Take the fold of this equation with $('x)^{-1}$ and use (2.80). We get

$$(2.82)\qquad u *('x)^{-1} + \pi^2\,\lambda u = f *('x)^{-1}.$$

Eliminate $u *('x)^{-1}$ between (2.81) and (2.82) and we finally obtain the solution of (2.81):

$$u = \frac{f+\lambda f *('x)^{-1}}{1+\pi^2\lambda^2}.$$

We shall close this section by deriving a formula which shows that caution must be used when performing algebraic manipulations on principal value integrals in more than one dimension. Consider the expression

$$\frac{1}{'(x+y)}\left[\frac{1}{'x}+\frac{1}{'y}\right].$$

Ordinary algebraic manipulations would indicate that this expression equals $('x)^{-1}('y)^{-1}$, but this is wrong. Using the result (page 88) that

$$\frac{1}{'x} = \lim_{\epsilon\to 0}\ \frac{x}{x^2+\epsilon^2},$$

we have

$$\frac{1}{'(x+y)}\left[\frac{1}{'x}+\frac{1}{'y}\right] = \lim \frac{1}{'(x+y)}\left[\frac{x}{x^2+\epsilon^2}+\frac{y}{y^2+\epsilon^2}\right]$$

$$(2.83)\qquad = \lim \frac{1}{'(x+y)}\ \frac{(x+y)(xy+\epsilon^2)}{(x^2+\epsilon^2)(y^2+\epsilon^2)} = \lim\left[\frac{xy}{(x^2+\epsilon^2)(y^2+\epsilon^2)}+\frac{\epsilon}{x^2+\epsilon^2}\cdot\frac{\epsilon}{y^2+\epsilon^2}\right]$$

$$= \frac{1}{'x}\cdot\frac{1}{'y}+\pi^2\delta(x)\delta(y),$$

again by the results of page 88. Formula (2.83) is useful in some quantum mechanical applications.

Problems

2.38 Evaluate $\int_{-\infty}^{\infty}\frac{\sin kx\,dx}{'(x-x_o)}$ and $\int_{-\infty}^{\infty}\frac{e^{-k|x|}}{'(x^2-x_o^2)}\,dx.$

2.39 Evaluate $\int_o^{\pi}\frac{d\theta}{\sin\theta}$ and $\int_o^{2\pi}\frac{d\theta}{a+b\cos\theta}$ if $|a/b|<1.$

2.40 Make the transformation $u = 2\tan(\theta/2)$ in the integrals of Problem 2.39 and then evaluate.

2.42 Evaluate $\int_o^{\infty}\frac{\sin kx\,dx}{x^2-x_o^2}$ if x_o has a small positive imaginary part. Compare the result with the integral $\int_o^{\infty}\frac{\sin kx\,dx}{'(x^2-x_o^2)}$.

2.42 Solve the integral equation

$$u(x)+\lambda\int_{-\infty}^{\infty}\frac{u(y)}{'(x-y)}\,dy = e^{-a|x|}.$$

Summary

The discussion of the present chapter has shown the utility and simplicity of the theory of distributions. It usually enables us to obtain in a simple way results which would be fairly complicated to obtain by customary methods. The position of the theory of distributions may be compared to that of the theory of complex variables in the study of acoustic or electromagnetic phenomena. These physical phenomena must be represented by real numbers. However, for the purpose of simplifying the calculations, we introduce complex numbers and at the end interpret these numbers to obtain a real physical answer. In the same way we shall henceforth, unless otherwise stated, assume that we are always dealing with the distributions defined by the functions, symbolic or ordinary, that appear in our calculations. If the final result of the calculation is a distribution defined by an ordinary function, the ordinary function will be the answer required. If the final result of the calculations is a distribution defined by a symbolic function, then the original problem does not have a solution in the usual sense. In this case, however, if the original problem is re-formulated so as to apply to a testing function, the answer would be equal to that obtained by applying the distribution defined by the symbolic function to the testing function.

We shall summarize some of the results of the present chapter in the following list of formulas:

1. If $f(x)$ is a piecewise continuous function with jumps of magnitude α_i at the points a_i ($i = 1, 2, \ldots k$) and if $f'(x)$ denotes the symbolic function for the derivative of the distribution defined by $f(x)$, while $\left[f'(x)\right]$ is the derivative, wherever it exists in the ordinary sense, of $f(x)$, then

$$f'(x) = \left[f'(x)\right] + \sum_{i=1}^{k} \alpha_i \, \delta(x-a_i).$$

Example. $\frac{d}{dx}$ signum $x = 2\delta(x)$.

2. If $f(x)$ is a continuous differentiable function with simple zeros at the points $r_1, r_2, \ldots, r_n$ then

$$\delta(f(x)) = \sum_{1}^{n} \frac{\delta(x-r_k)}{|f'(r_k)|}.$$

Example. $\delta(x^2-a^2) = \frac{1}{2a} \cdot \left[\delta(x+a)+\delta(x-a)\right]$.

3. $x^m \delta^{(m+n)}(x) = (-)^m \frac{(m+n)!}{n!} \delta^{(n)}(x)$, $n \geq 0$, m an integer.

4. $x^{m+n} \delta^{(m)}(x) = 0$.

5. $f(x)\delta(x) = f(0)\delta(x)$; $g(x)\delta'(x) = -g'(0)\delta(x)+g(0)\delta'(x)$.

6. $$\int_0^\infty \frac{\phi(x)}{('x)^m}\,dx = \lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{\phi(x)}{x^m}dx + \frac{\phi(0)\epsilon^{1-m}}{0!(1-m)} + \frac{\phi'(0)\epsilon^{2-m}}{1!(2-m)} + \ldots + \frac{\phi^{(m-1)}(0)\log\epsilon}{(m-1)!}\right].$$

m integral

7. $$\int_0^\infty \frac{\phi(x)}{('x)^p}\,dx = \lim_{\epsilon\to 0}\left[\int_0^\infty \frac{\phi(x)}{x^p}\,dx + \frac{\phi(0)\epsilon^{1-p}}{0!(1-p)} + \ldots + \frac{\phi^{(m-1)}(0)}{(m-1)!}\,\frac{\epsilon^{m-p}}{m-p}\right].$$

p non-integral; m is the largest integer contained in p.

8. $$\frac{d\log|x|}{dx} = \frac{1}{'x}, \quad \frac{d\log x}{dx} = \frac{1}{'x} - i\pi\delta(x).$$

9. $$\frac{d}{dx}\left[('x)^{-m}H(x)\right] = -m('x)^{-m-1}H(x) + (-)^m\frac{\delta^{(m)}(x)}{m!}.$$

$$\frac{d}{dx}\left[('y)^{-p}H(x)\right] = -p('y)^{-p-1}H(x).$$

$$\frac{d}{dx}('y)^{-m} = -m('x)^{-m-1}.$$

$$x^q('x)^p = ('x)^{p+q}.$$

10. $$\int_0^\infty ('x)^{p-1}e^{-ax}dx = \frac{\Gamma(p)}{('a)^p}.$$

11. $u = \dfrac{H(x)}{(n-2)('x)^{n-2}}$ is a solution of $(x^{n-1}u')' = -\delta$.

12. $$\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^2+x^2} = \pi\delta(y), \quad \lim_{\epsilon\to 0}\frac{x}{\epsilon^2+x^2} = \frac{1}{'x}.$$

13. $$\int_{-\infty}^\infty k^n e^{ikx}dk = 2\pi(-i)^n\delta^{(n)}(x).$$

14. $$\int_0^\infty k^n e^{ikx}dk = \pi(-i)^n\delta(x) + \frac{i^{n+1}n!}{('x)^{n+1}}.$$

15. $$\int_{-\infty}^\infty \operatorname{sgn} k\, e^{ikx}dk = \frac{2i}{'y}.$$

16. $$\delta(y)\delta(y) = \frac{\delta(r)}{2\pi r}; \quad r^2 = x^2+y^2.$$

17. $$\delta(x)\delta(y)\delta(z-z_0) = \frac{\delta(r-r_0)\delta(0)}{\pi r^2}.$$

$$\delta(x)\delta(y)\delta(z) = \frac{\delta(r)}{\pi r^2}.$$

$x^2 = x^2+y^2$, $z = r\sin\theta$, $z_0 = r_0$.

Date Due

OCT 26 1955			
~~NOV 31 '58~~	OCT [illegible] '62		
DEC 1 '58	OCT [illegible] '62		
[illegible] 16 '58	MAY 22 64		
DEC 29 '58	FEB 10 '68		
JAN 18 '60	APR 21 1969		
FEB 18 '60	[illegible]		
MAR 19 '60	JUL 23 1969		
MAR 14 '60	AUG [illegible] 1970		
MAY 2 60	[illegible]		
	MAR 2 1971		
[illegible]	MAY 7 1973		
[illegible]	NOV [illegible] 1979		
JUN 3 '62			
JUN 22 '62			
MAR 12 '62			
AUG 15 '62			
	PRINTED IN U. S. A.		

NYU- 47

Friedman, B.

Techniques of appl. math